GESELLSCHAFT für
ANGEWANDTE MATHEMATIK und MECHANIK e.V.
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Structure-preserving methods for port-Hamiltonian flexible multibody systems

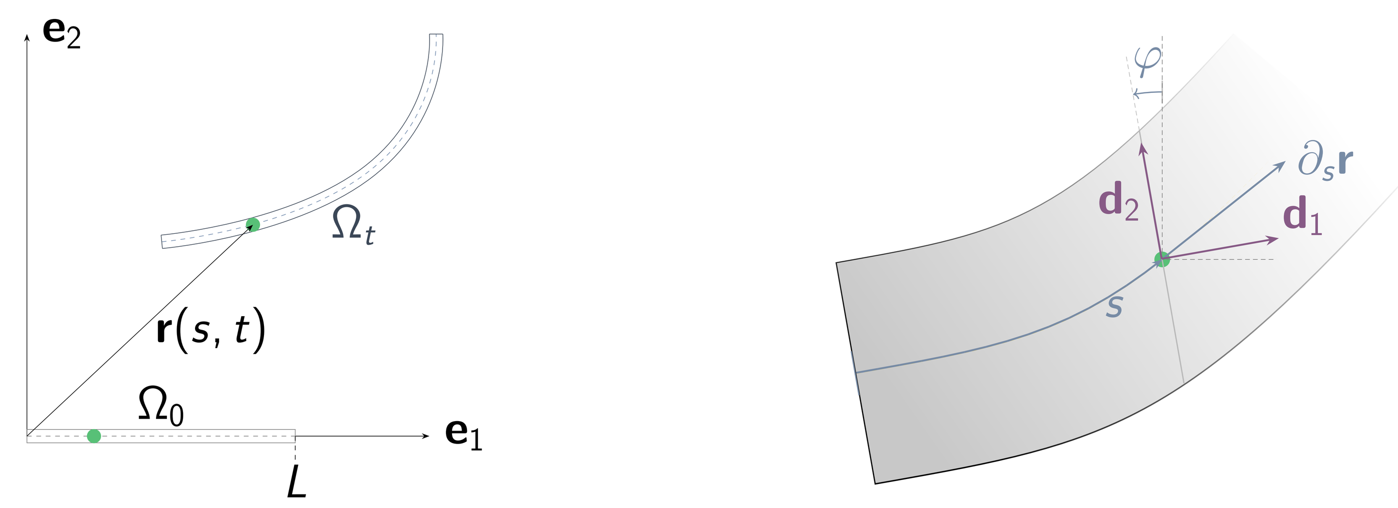
Abstract

Problem: Classical methods for flexible multibody systems (MBS) may suffer from inaccurate kinematics, numerical instabilities or physical inconsistencies.

Strategy: Combine the following ingredients

- **Geometrically exact beams (GEB)**: Large rotations and displacements with objective strain measures
- **Port-Hamiltonian systems (PHS)**: Energy-based, modular modelling framework
- **Structure-preserving discretization**: Robust simulations with guaranteed physical consistency

Planar geometrically exact beams



Consider a 1D beam with $s \in \Omega = [0, L]$, $t \in [0, T]$ and nonlinear kinematics described by positions $\mathbf{r}(s,t) \in \mathbb{R}^2$ and angle $\varphi(s,t)$ inducing the rotation matrix $\mathbf{\Lambda}(\varphi) \in SO(2)$ for the orientation of cross-sections remaining planar during motion. We assume objective material strains $\mathbf{\Gamma} = \mathbf{\Lambda}^{\mathsf{T}}\partial_s \mathbf{r} - \mathbf{e}_1$ for dilatation and shear as well as the curvature $\kappa = \partial_s \varphi$. The dynamics are governed by the kinematical evolution equations

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \omega = \dot{\varphi}, \quad \dot{\kappa} = \partial_s \omega, \quad \dot{\mathbf{\Gamma}} = \dot{\mathbf{\Lambda}}(\varphi,\omega)^{\mathsf{T}}\partial_s \mathbf{r} + \mathbf{\Lambda}(\varphi)^{\mathsf{T}}\partial_s \mathbf{v}$$

together with balance equations of momentum

$$\rho A \dot{\mathbf{v}} = \partial_s(\mathbf{\Lambda N}) + \mathbf{n}_{\mathrm{ext}}, \qquad \rho I \dot{\omega} = \partial_s M + \partial_s \mathbf{r} \times (\mathbf{\Lambda N}) + m_{\mathrm{ext}},$$

where (possibly hyperelastic) constitutive laws $\mathbf{N} = \partial_{\mathbf{\Gamma}} W$, $M = \partial_\kappa W$ are considered. The initial-boundary-value problem is completed by appropriate boundary and initial conditions.

Port-Hamiltonian formulation [1]

Procedure:

1) Mixed, structure-preserving finite element approximation with C^0-continuous, linear ansatz for $(\mathbf{r}, \mathbf{v}, \varphi, \omega)$ and discontinuous, piecewise constant ansatz for $(\kappa, \mathbf{\Gamma})$
2) Nodal unknowns as state (or *energy-variables*) $\mathbf{x} = (\hat{\mathbf{r}}, \hat{\mathbf{v}}, \hat{\varphi}, \hat{\omega}, \hat{\kappa}, \hat{\mathbf{\Gamma}})$
3) Approximate total energy $\mathcal{H}$ (termed *Hamiltonian*)

The port-Hamiltonian form of governing equations

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{J}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \mathbf{B}\mathbf{u} \quad \text{with} \quad \mathbf{E}^{\mathsf{T}}\mathbf{z}(\mathbf{x}) = \nabla\mathcal{H}(\mathbf{x}), \\ \mathbf{y} &= \mathbf{B}^{\mathsf{T}}\mathbf{z}(\mathbf{x}), \end{aligned}$$

with $\mathbf{E} = \mathbf{E}^{\mathsf{T}}$ and $\mathbf{J} = -\mathbf{J}^{\mathsf{T}}$, immediately induces the power balance equation

$$\dot{\mathcal{H}} = \nabla\mathcal{H}(\mathbf{x})^{\mathsf{T}}\dot{\mathbf{x}} = \mathbf{z}^{\mathsf{T}}\mathbf{E}\dot{\mathbf{x}} = \mathbf{z}^{\mathsf{T}}\mathbf{J}(\mathbf{x})\mathbf{z} + \mathbf{z}^{\mathsf{T}}\mathbf{B}\mathbf{u} = \mathbf{y}^{\mathsf{T}}\mathbf{u}.$$

- Rigid MBS in index-2 form with velocity-level constraints fit into the very same formulation with $\mathbf{x} = (\mathbf{q}, \mathbf{v}, \boldsymbol{\lambda})$ yielding a non-invertible $\mathbf{E}$.
- Easily extendable to dissipative effects (see [3]) and readily available for consistent coupling with other port-Hamiltonian models (possibly from other physical domains).

Consistent time discretization [2]

Consider the following second-order accurate, one-step method with $h = t^{n+1} - t^n$, where the discrete gradient $\overline{\nabla}\mathcal{H}$ appears implicitly in the constitutive relation and $\bar{\mathbf{J}} = -\bar{\mathbf{J}}^{\mathsf{T}}$ denotes a consistent approximation of $\mathbf{J}$.

$$\begin{aligned} \mathbf{E}(\mathbf{x}^{n+1} - \mathbf{x}^n) &= h\bar{\mathbf{J}}(\mathbf{x}^n, \mathbf{x}^{n+1})\mathbf{z}^{n,n+1} + h\mathbf{B}\mathbf{u}^{n,n+1}, \\ \mathbf{y}^{n,n+1} &= \mathbf{B}^{\mathsf{T}}\mathbf{z}^{n,n+1}, \end{aligned} \quad \text{with} \quad \mathbf{E}^{\mathsf{T}}\mathbf{z}^{n,n+1} = \overline{\nabla}\mathcal{H}(\mathbf{x}^n, \mathbf{x}^{n+1}).$$

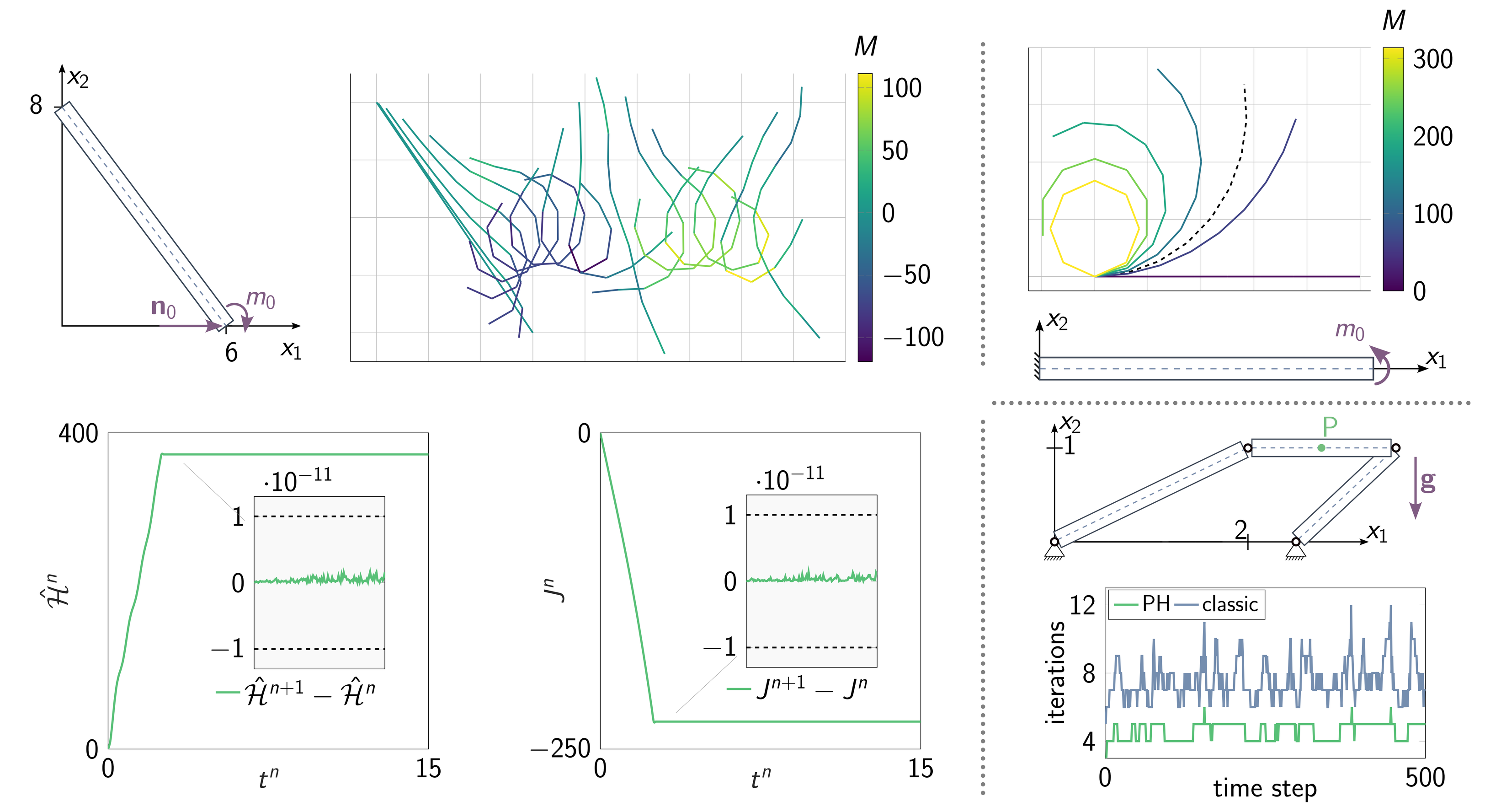
- Consistent discrete-time power balance

$$\mathcal{H}(\mathbf{x}^{n+1}) - \mathcal{H}(\mathbf{x}^n) = \overline{\nabla}\mathcal{H}(\mathbf{x}^n, \mathbf{x}^{n+1})^{\mathsf{T}}(\mathbf{x}^{n+1} - \mathbf{x}^n) = h\,(\mathbf{y}^{n,n+1})^{\mathsf{T}}\mathbf{u}^{n,n+1}.$$

- The DAE case (non-invertible $\mathbf{E}$) requires additional considerations. For semi-explicit DAEs with $\mathbf{E} = \mathrm{diag}(\mathbf{E}_{11}, \mathbf{0})$ the time-discretization is especially insightful, see [2].

Numerical results

The experiments show that our approach yields numerically robust, stable and locking-free simulations with guaranteed physical consistency even with large deformations.



References

[1] P. L. Kinon, P. Betsch, and S. R. Eugster. “Energy-Momentum-Consistent Simulation of Planar Geometrically Exact Beams in a Port-Hamiltonian Framework”. In: *Multibody Syst. Dyn. (under revision)* (2025). DOI: `10.21203/rs.3.rs-6255031/v1`

[2] P. L. Kinon, R. Morandin, and P. Schulze. “Discrete gradient methods for port-Hamiltonian differential-algebraic equations”. In: *preparation* (2025)

[3] P. L. Kinon, T. Thoma, P. Betsch, and P. Kotyczka. “Generalized Maxwell Viscoelasticity for Geometrically Exact Strings: Nonlinear Port-Hamiltonian Formulation and Structure-Preserving Discretization”. In: *IFAC-PapersOnLine* 58.6 (2024), pp. 101–106. DOI: `10.1016/j.ifacol.2024.08.264`

Related talk: Thursday, April 10, 16:30: S04.08 (Structural mechanics)
PLK, P. Betsch, S. R. Eugster:
“*Geometrically exact planar beam dynamics: Port-Hamiltonian modeling and structure-preserving discretization*”

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  - (Flexible) multibody dynamics
  - Port-Hamiltonian systems
  - Structure-preserving discretization methods