GESELLSCHAFT für
ANGEWANDTE MATHEMATIK und MECHANIK e.V.
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Port-Hamiltonian Cosserat rod dynamics

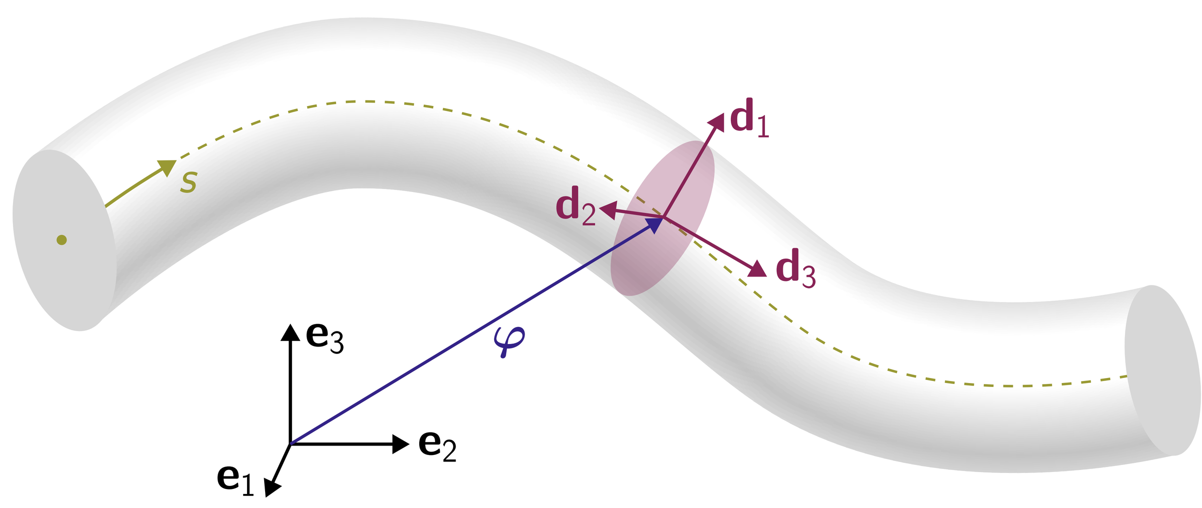
Abstract

Problem: Classical methods for highly flexible slender structures may suffer from approximated kinematics, numerical instabilities or physical inconsistencies.

Strategy: Combine the following ingredients

- **Cosserat rod model**: Accurate description of large rotations and displacements
- **Port-Hamiltonian (PH) systems**: Energy-based, well-structured, modular modeling framework
- **Structure-preserving discretization**: Robust simulations with guaranteed physical consistency

PH Cosserat rods using directors



Consider a beam with $s \in \Omega = [0, L]$ and time $t \in [0, T]$. The nonlinear kinematics are described by the centerline position $\varphi \in \mathbb{R}^3$ and the directors $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) \in \mathbb{R}^9$, which form a cross-section-fixed orthonormal frame inducing the rotation matrix $\mathbf{R}(\mathbf{d}) \in SO(3)$. The configuration is given by $\mathbf{q} = (\varphi, \mathbf{d}) \in \mathbb{R}^{12}$. We assume material strain measures $\mathbf{\Gamma}(\mathbf{q}, \mathbf{q}_{,s})$ for dilatation and shear, and $\mathbf{K}(\mathbf{q}, \mathbf{q}_{,s})$ for curvature and torsion, with conjugate stress resultants $\mathbf{N}$ and $\mathbf{M}$. Independent velocities $\mathbf{v} = (\mathbf{v}_\varphi, \mathbf{v}_d)$ are introduced such that $\dot{\mathbf{q}} = \mathbf{v}$. The dynamics are governed by the balance equations of momentum

$$\rho A \dot{\mathbf{v}}_\varphi = \partial_s(\mathbf{R}\mathbf{N}), \qquad \mathbf{M}_\rho \dot{\mathbf{v}}_d = -\mathbf{J}_\mathbf{N}\mathbf{N} - \mathcal{J}_\mathbf{M}\mathbf{M} - \mathbf{G}_d^\mathsf{T}\boldsymbol{\lambda},$$

together with 6 orthonormality conditions on velocity level $\partial_\mathbf{d}\mathbf{g}(\mathbf{d})\mathbf{v}_d = \mathbf{0}$.

The independent stress variables $\mathbf{N}$ and $\mathbf{M}$ are related to the kinematics through **compliance equations** in rate form

$$\mathbf{C}_N \dot{\mathbf{N}} = \mathbf{R}^\mathsf{T}\partial_s \mathbf{v}_\varphi + \mathbf{J}_\mathbf{N}^\mathsf{T}\mathbf{v}_d, \qquad \mathbf{C}_M \dot{\mathbf{M}} = \mathcal{J}_\mathbf{K}\mathbf{v}_d.$$

Here, $\mathbf{C}_N$ and $\mathbf{C}_M$ are diagonal compliance matrices, whose elements may be set to zero to easily account for **inextensibility** or **shear-rigidity**.

The initial-boundary-value problem exhibits a PH structure and is completed by appropriate boundary and initial conditions.

Mixed FEM

The continuous PH system is transformed into a discrete PH system by a structure-preserving **mixed finite element method** (FEM). Hereto, we choose a C^0-continuous, quadratic ansatz for $(\mathbf{q}, \mathbf{v})$ and discontinuous, piecewise linear approximations for $(\mathbf{N}, \mathbf{M})$. The multipliers $\boldsymbol{\lambda}$ are approximated collocation-wise. The FE unknowns are comprised in the state variables $\mathbf{x}$ and the total energy $\mathcal{H}$ is termed Hamiltonian.

The **port-Hamiltonian** (PH) system is given by

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{B}^\mathsf{T}\mathbf{z}(\mathbf{x}),$$

where $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\mathsf{T}$ and $\mathbf{E}^\mathsf{T}\mathbf{z}(\mathbf{x}) = \nabla\mathcal{H}(\mathbf{x})$ is satisfied. By design, this induces the **energy balance equation**

$$\dot{\mathcal{H}} = \nabla\mathcal{H}(\mathbf{x})^\mathsf{T}\dot{\mathbf{x}} = \mathbf{z}^\mathsf{T}\mathbf{E}\dot{\mathbf{x}} = \mathbf{z}^\mathsf{T}\mathbf{J}(\mathbf{x})\mathbf{z} + \mathbf{z}^\mathsf{T}\mathbf{B}\mathbf{u} = \mathbf{y}^\mathsf{T}\mathbf{u}.$$

- Mixed formulation is **locking-free** and **objective**,
- **Easily extendable** to dissipative effects using a generalized Maxwell model, and the coupling with other PH models (rigid bodies, thermomechanical systems, etc.).
- Enjoys the **insightful mathematical structure** of PH systems.

Structure-preserving time integration

The PH structure guides the design of an **energy–momentum-consistent** time integration scheme. Given a quadratic $\mathcal{H}$, the midpoint rule can be used to obtain a **second-order accurate** method. We denote $h = t^{n+1} - t^n$ and $\mathbf{x}^{n+1/2} = \frac{1}{2}(\mathbf{x}^{n+1} - \mathbf{x}^n)$.

$$\mathbf{E}(\mathbf{x}^{n+1} - \mathbf{x}^n) = h\mathbf{J}(\mathbf{x}^{n+1/2})\mathbf{z}(\mathbf{x}^{n+1/2}) + h\mathbf{B}\mathbf{u}^{n+1/2},$$
$$\mathbf{y}^{n+1/2} = \mathbf{B}^\mathsf{T}\mathbf{z}(\mathbf{x}^{n+1/2}).$$

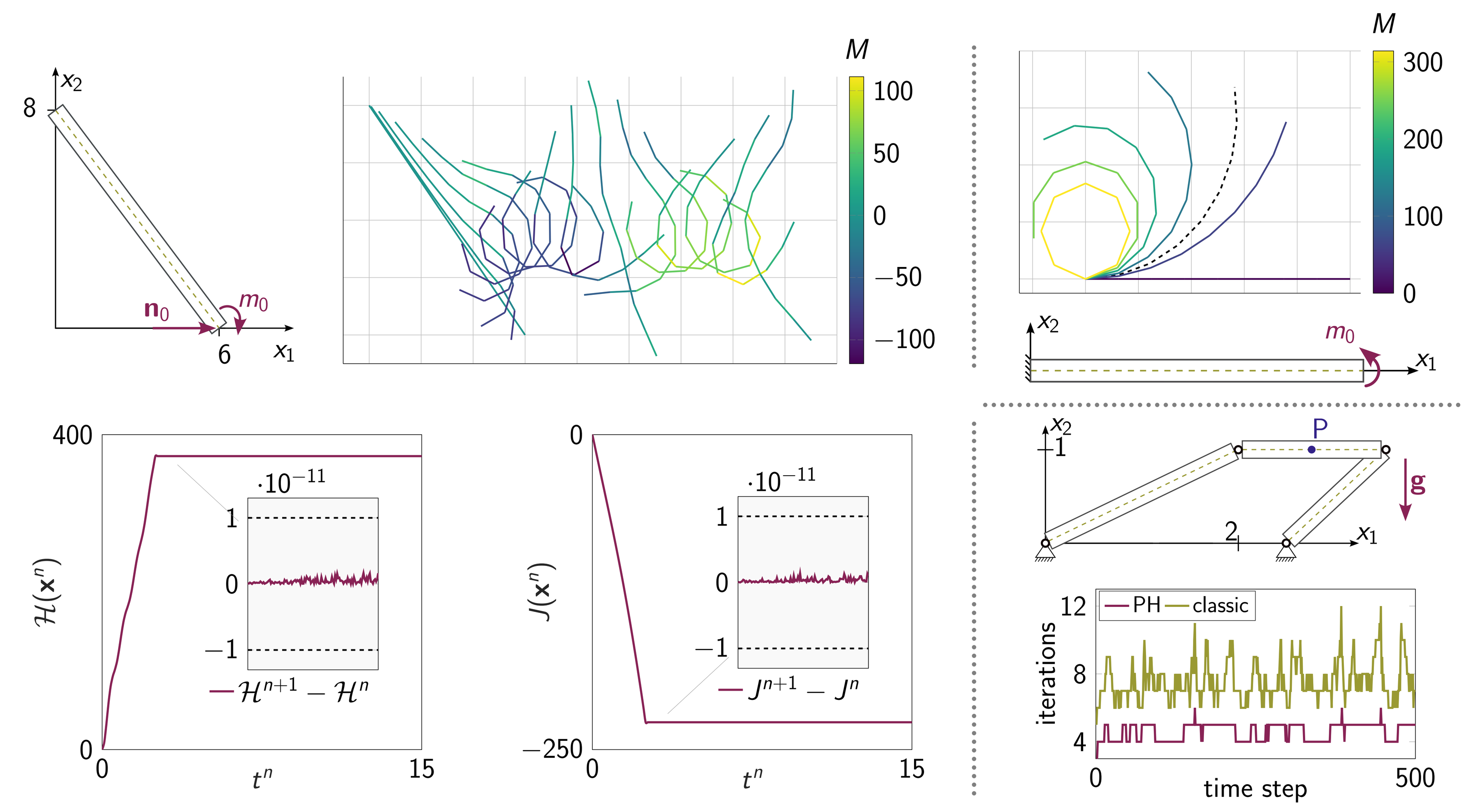
- Consistent discrete-time energy balance

$$\mathcal{H}(\mathbf{x}^{n+1}) - \mathcal{H}(\mathbf{x}^n) = \nabla\mathcal{H}(\mathbf{x}^{n+1/2})^\mathsf{T}(\mathbf{x}^{n+1} - \mathbf{x}^n) = h\,(\mathbf{y}^{n+1/2})^\mathsf{T}\mathbf{u}^{n+1/2}.$$

- For hyperelastic materials, **discrete gradient methods** can be designed, see [3].

Results & conclusion

The experiments show that our approach yields accurate and locking-free simulations with energy–momentum consistency even for large deformations. The method is **remarkably robust**, allowing much larger time steps than alternative Cosserat rod frameworks.



Numerical experiments in 3D can be found in [1]. This includes comparisons with analytic solutions and the literature, the simulation of viscoelastic beams and the **actuation of a soft robotic arm**, using **tendons** and **pneumatic chambers**.

References

[1] P. L. Kinon, S. R. Eugster, and P. Betsch. "Mixed formulation and structure-preserving discretization of Cosserat rod dynamics in a port-Hamiltonian framework". In: *Computer Methods in Applied Mechanics and Engineering (under review)* (2026). DOI: 10.48550/arXiv.2512.19408

[2] P. L. Kinon, P. Betsch, and S. R. Eugster. "Energy-momentum-consistent simulation of planar geometrically exact beams in a port-Hamiltonian framework". In: *Multibody System Dynamics* (2025). DOI: 10.1007/s11044-025-10087-9

[3] P. L. Kinon, R. Morandin, and P. Schulze. "Discrete gradient methods for port-Hamiltonian differential-algebraic equations". In: *Applied Numerical Mathematics* 223 (2026), pp. 45–75. DOI: 10.1016/j.apnum.2025.12.006

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"*Structure-preserving discretization of Cosserat rod dynamics in a mixed DAE-framework*"

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- Research interests:
  - (Flexible) multibody dynamics
  - Port-Hamiltonian systems
  - Structure-preserving discretization methods