KIT (Germany): Institute of Mechanics
TU/e (The Netherlands): Dynamics & Control

Energy-Momentum-Consistent Simulation of Planar Geometrically Exact Beams as PHS

Philipp L. Kinon (KIT) | Peter Betsch (KIT) | Simon R. Eugster (TU/e)

Motivation

Flexible multibody systems (MBS)

Wishlist for a formulation:

- Allow finite rotations and displacements
- Avoid numerical *locking*
- Robust and objective
- Energy-momentum (EM) consistent discretization

PHS modeling ideally suited:

- Structured formalism: Energetic consistency
- Intrinsically mixed
- Modular interconnection
- Easily extendable: DAEs and multiphysics

Problem description

- Geometrically exact beams (GEB)
  - IBVP on a 1D continuum with $s \in \Omega = [0, L]$
- Nonlinear 2D kinematics:
  - Position $\boldsymbol{r}(s,t) \in \mathbb{R}^2$ and orientation $\varphi(s,t)$
  - Rotation matrix

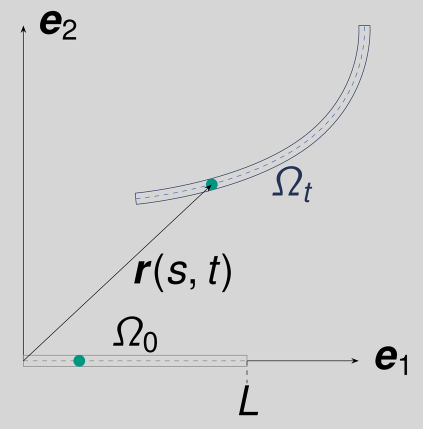
$$\boldsymbol{\Lambda}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_1(\varphi) & \boldsymbol{d}_2(\varphi) \end{bmatrix} \in SO(2)$$

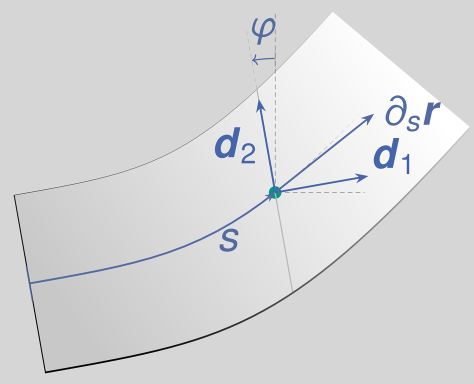
  - Strains (dilatation, shear and curvature)

$$\boldsymbol{\Gamma} = \boldsymbol{\Lambda}^\top \partial_s \boldsymbol{r} - \boldsymbol{e}_1 \in \mathbb{R}^2 \qquad \kappa = \partial_s \varphi$$

  - Internal forces, moments and strain energy function

$$\boldsymbol{n} = \boldsymbol{\Lambda N} \quad m = M \quad \boldsymbol{N} = \partial_{\boldsymbol{\Gamma}} W \quad M = \partial_\kappa W$$





- Kinematic evolution

$$\partial_t \boldsymbol{r} = \boldsymbol{v} \quad \partial_t \varphi = \omega \quad \partial_t \kappa = \partial_s \omega \quad \partial_t \boldsymbol{\Gamma} = \omega \boldsymbol{\Lambda}^\top \boldsymbol{S} \partial_s \boldsymbol{r} + \boldsymbol{\Lambda}^\top \partial_s \boldsymbol{v}$$

- Balance equations with external loads $(\overline{\boldsymbol{n}}, \overline{m})$

$$\rho A \partial_t \boldsymbol{v} = \partial_s \boldsymbol{n} + \overline{\boldsymbol{n}} \qquad \rho I \partial_t \omega = \partial_s M + \partial_s \boldsymbol{r}^\top \boldsymbol{S} \boldsymbol{n} + \overline{m}$$

PH modeling and structure-preserving discretization in space and time

PH formulation

- State (energy-variables)

$$\boldsymbol{x} = (\boldsymbol{r}, \varphi, \boldsymbol{v}, \omega, \boldsymbol{\Gamma}, \kappa)$$

- Hamiltonian

$$\mathcal{H}(\boldsymbol{x}) = \int_\Omega \left(\frac{1}{2} \rho A \boldsymbol{v}^\top \boldsymbol{v} + \frac{1}{2} \rho I \omega^2 + W(\boldsymbol{\Gamma}, \kappa) \right) \mathrm{d}s$$

- Local PH equations (where $\mathcal{J} = -\mathcal{J}^*$)

$$\boxed{\begin{aligned} \mathcal{E} \partial_t \boldsymbol{x} &= \mathcal{J}(\boldsymbol{x}) \boldsymbol{z}(\boldsymbol{x}) + \mathcal{B} \boldsymbol{u} \quad \text{with} \quad \mathcal{E}^\top \boldsymbol{z}(\boldsymbol{x}) = \delta_x \mathcal{H} \\ \boldsymbol{y} &= \mathcal{B}^\top \boldsymbol{z}(\boldsymbol{x}) \end{aligned}}$$

- Power balance

$$\partial_t \mathcal{H} = \left[\boldsymbol{v}^\top \boldsymbol{n} + \omega M \right]_{\partial\Omega} + (\boldsymbol{u}, \boldsymbol{y})_\Omega = \boldsymbol{u}_\partial^\top \boldsymbol{y}_\partial + (\boldsymbol{u}, \boldsymbol{y})_\Omega$$

Mixed FEM

- 2-node elements, ansatz functions:
  - C^0-continuous, linear for $(\boldsymbol{r}, \boldsymbol{v}, \varphi, \omega)$
  - discontinuous and constant for $(\boldsymbol{\Gamma}, \kappa)$
- Discrete state $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{r}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\kappa}})$ and

$$\hat{\mathcal{H}}(\hat{\boldsymbol{x}}) = \frac{1}{2} \hat{\boldsymbol{v}}^\top \rho A \boldsymbol{M}_2 \hat{\boldsymbol{v}} + \frac{1}{2} \hat{\boldsymbol{\omega}}^\top \rho I \boldsymbol{M}_1 \hat{\boldsymbol{\omega}} + \hat{W}(\hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\kappa}})$$

- Resulting finite-dimensional PHS

$$\boxed{\begin{aligned} \boldsymbol{E} \partial_t \hat{\boldsymbol{x}} &= \boldsymbol{J}(\hat{\boldsymbol{x}}) \hat{\boldsymbol{z}}(\hat{\boldsymbol{x}}) + \boldsymbol{B} \hat{\boldsymbol{u}} \quad \text{with} \quad \boldsymbol{E}^\top \hat{\boldsymbol{z}}(\hat{\boldsymbol{x}}) = \nabla_{\hat{x}} \hat{\mathcal{H}} \\ \hat{\boldsymbol{y}} &= \boldsymbol{B}^\top \hat{\boldsymbol{z}}(\hat{\boldsymbol{x}}) \end{aligned}}$$

- Power balance: $\partial_t \hat{\mathcal{H}} = \hat{\boldsymbol{u}}^\top \hat{\boldsymbol{y}}$

Midpoint-based integration

- 2nd-order one-step schemes with $\boldsymbol{x}_n \approx \boldsymbol{x}(t_n)$, h = const. and $\hat{\boldsymbol{x}}_{n+1/2} = \frac{1}{2}(\hat{\boldsymbol{x}}_{n+1} + \hat{\boldsymbol{x}}_n)$ given by

$$\boxed{\begin{aligned} \boldsymbol{E}(\hat{\boldsymbol{x}}_{n+1} - \hat{\boldsymbol{x}}_n) &= h \boldsymbol{J}(\hat{\boldsymbol{x}}_{n+1/2}) \hat{\boldsymbol{z}}_{n,n+1} + h \boldsymbol{B} \hat{\boldsymbol{u}}_{n,n+1} \\ \hat{\boldsymbol{y}}_{n,n+1} &= \boldsymbol{B}^\top \hat{\boldsymbol{z}}_{n,n+1} \end{aligned}}$$

- Discrete-time PHS by discrete gradient

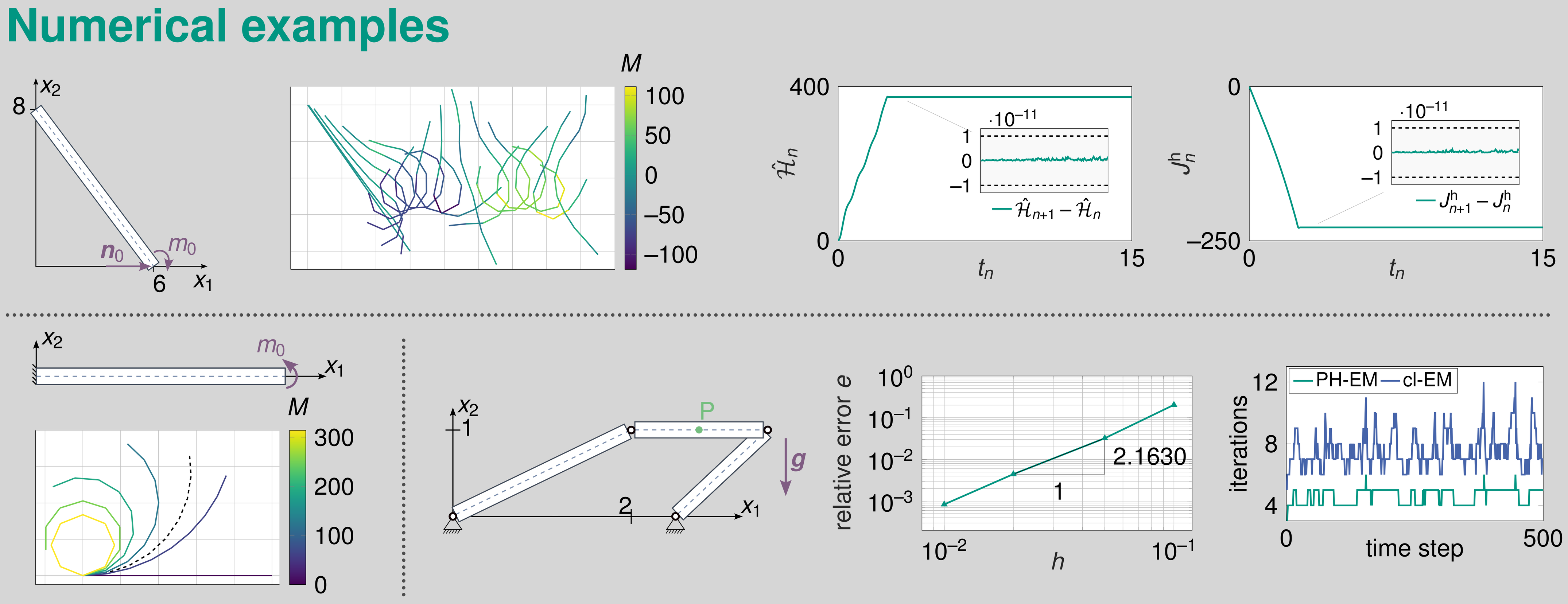
$$\boxed{\boldsymbol{E}^\top \hat{\boldsymbol{z}}_{n,n+1} = \overline{\nabla} \hat{\mathcal{H}}(\hat{\boldsymbol{x}}_n, \hat{\boldsymbol{x}}_{n+1})}$$

- Consistent power balance

$$\hat{\mathcal{H}}_{n+1} - \hat{\mathcal{H}}_n = h\, \hat{\boldsymbol{u}}_{n,n+1}^\top \hat{\boldsymbol{y}}_{n,n+1}$$

- Consistent angular momentum balance

Numerical examples



Take-Home message

We achieved a combination of two state-of-the-art modeling approaches:

- **GEB**: Finite displacements and rotations ✓
- **PHS**: Mixed, objective and robust approach yields EM-scheme in a natural way and avoids locking ✓

Together they provide a powerful tool for accurate simulations of flexible MBS

Future research directions:

- Interconnection with other PH MBS models
- Extension to 3D (rotation parametrization)

References

1. PLK, PB, SRE (2025): **Energy-momentum-consistent simulation of planar geometrically exact beams in a port-Hamiltonian framework** (submitted), doi: 10.21203/rs.3.rs-6255031/v1
2. PLK, T. Thoma, PB, P. Kotyczka (2024): Generalized Maxwell viscoelasticity for geometrically exact strings. In: IFAC-PapersOnLine 58.6, doi: 10.1016/j.ifacol.2024.08.264
3. PLK, T. Thoma, PB, P. Kotyczka (2023): Discrete nonlinear elastodynamics in a port-Hamiltonian framework. In: PAMM 23.3, doi: 10.1002/pamm.202300144

Philipp L. Kinon – philipp.kinon@kit.edu
Doctoral researcher
Web: www.ifm.kit.edu/english/14_5490.php
RG: www.researchgate.net/profile/Philipp-Kinon