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Port-GENERIC Models and Their Couplings for Thermo-Visco-Elastic Systems

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Abstract

This paper addresses port-based models for open multi-physical systems with reversible and irreversible couplings in the framework of the so-called GENERIC (General Equation for Non-Equilibrium Reversible–Irreversible Coupling) formalism. For this purpose, we introduce the term "port-GENERIC", which makes it possible to describe multiphysical systems that arise from the interconnection of subsystems, which are themselves expressed in port-GENERIC form. We start from the special formulation of GENERIC according to Mielke (2011), which allows the deliberate choice of thermal variables. As is well known, the Poisson matrix assumes a canonical form if the subsystem entropies are chosen as thermal states, corresponding to the interconnection matrix of a quasi-Hamiltonian formulation. It is visualized by the topology of classical bond graphs. We show how the Onsager matrix with the choice of internal energies as thermal states is related to the components and topology of entropy bond graphs, i.e., bond graphs for the entropy flows. Using a didactic example of a thermo-visco-elastic model problem, we show the composition of port-GENERIC models via different kinds of couplings: purely reversible interconnections, couplings of reversible with irreversible subsystems and thermal interfaces.

Keywords: reversible–irreversible couplings; GENERIC; bond graphs; irreversible systems; thermo-visco-elasticity; port-based modeling of control systems

1. Introduction

The scope of this contribution is models for *open* multi-physical dynamical systems with *reversible* and *irreversible* couplings. It is shown how such models can be derived systematically from the interconnection of subsystems describing storage and conversion of different energy forms, including heat. While such interconnections must be *power-conserving*, i.e., they must respect the first law of thermodynamics, the subsystem and interface descriptions must display both the first and second law; in particular, *entropy production* or *entropy flows* related to irreversible phenomena and heat supply must be consistently expressed by the models.

The *metriplectic* [1,2] or *GENERIC* [3] formulations of a dynamical system are based on an *additive splitting* of reversible and irreversible contributions, which are derived from the *total energy* and the *total entropy* of a (sub-)system. The term *metriplectic* refers to both the symplectic and the metric-dependent nature of Hamiltonian and dissipative dynamics, while *GENERIC* stands for *General Equation for Non-Equilibrium Reversible–Irreversible*

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Coupling, introduced for non-equilibrium thermodynamics. Mielke [4] proposed a *special form* of the GENERIC matrices, which encodes the *free choice* of thermodynamic variables through a decomposition of the GENERIC matrices with the Jacobians of the corresponding state transformations. The GENERIC formulation for *open* systems has been discussed, e.g., in [5,6]. Structure-preserving discretization of GENERIC models allows the derivation of *energy-momentum-entropy*-consistent space-time discretization schemes [7–9].

Port-Hamiltonian models [10,11] are a port- and energy-based description of control systems. Their definition was heavily inspired by the concept of *bond graphs*, a graphical representation of energy flows in interconnected multi-physical systems [12,13]. So-called *irreversible port-Hamiltonian systems* (iPHS) [14] extend the port-Hamiltonian formalism towards irreversible phenomena by defining a *Poisson bracket between energy and entropy*, which leads to a model that formally has port-Hamiltonian structure but is *modulated* by the co-state variables. See also the recent works [15] on the interconnection of such models, [16] on their structure-preserving time integration, and [17] for the use in optimal control.

Very recent geometric formulations of reversible–irreversible systems are based on a metriplectic 4-bracket [18] or generating functions on smooth manifolds [19].

While the different representations must display the same physics and can be transformed one into the other, a main feature of (port-)GENERIC is that energy and entropy balances follow in a structurally identical way from the additive splitting of reversible and irreversible dynamics with the two potentials energy and entropy. This is different in iPHS, where only the gradient of the total energy appears directly on the right-hand side of the state differential equations; the entropy balance follows from a Poisson bracket between energy and entropy.

In this context, this paper seeks to combine the simple structure and versatility of the special GENERIC formulation with the clarity of couplings in port-Hamiltonian systems and the graphical intuition of the language of bond graphs. More specifically, the transformations between thermal variables derived in [4], in particular between the subsystem entropies and energies, help to visualize the irreversible phenomena in so-called *entropy bond graphs*. Our contributions are as follows.

- The first contribution of this paper is to propose a class of *port-GENERIC* systems, which feature ports of different kinds: *purely reversible ports*, which, if connected with another reversible port, carry a bidirectional energy flow; *irreversible ports*, where energy degradation (to heat) occurs, and where the energy flow is unidirectional; and lastly, thermal interfaces, which connect subsystems via their thermal variables, e.g., through heat conduction.
- As a second contribution, we exploit the transformation between entropies and internal energies as subsystem thermal variables to relate the matrices of port-GENERIC to bond graphs for both energy and entropy flows. In particular, we characterize the structure of the canonical *Onsager* matrix $\mathbf{K}_0 = \mathbf{K}_0^\top$ if internal energies are chosen as thermal states for the subsystems, but also for irreversible interconnections of subsystems. It is remarkable that the entries of the corresponding matrices can be related to the topology and the *modulated resistors* (MR elements) of the *entropy bond graphs*. These represent, in contrast to the classical (energy) bond graphs, entropy generation, routing and storage for the irreversible system part.
- As a third contribution, we discuss the didactic example of a finite-dimensional thermo-visco-elastic system, thus illustrating the concepts.

This paper is organized as follows: In Section 2, we recall some background on bond graphs, GENERIC and its special form. In Section 3, we propose a formulation of lumped-parameter port-GENERIC systems, which feature different reversible and irreversible couplings. Section 4 presents a lumped example of a thermo-visco-elastic system as a model

problem to illustrate the port-GENERIC modeling of subsystems and their interconnection through the different types of ports. For all subsystems, the energy and entropy bond graphs are drawn, and the relations of the latter to the canonical Onsager matrices $\mathbf{K}_0$ are clearly pointed out. Section 5 concludes this paper with an outlook on ongoing and future work.

Remark 1. *This article extends and refines our conference paper [20], where we illustrated our main ideas with a purely academic example, and discusses the particular case where a subset of the non-thermal variables contributes* exclusively *to irreversible phenomena, leading to a block-diagonal canonical Onsager matrix* $\mathbf{K}_0$*. The didactic example of this paper illustrate the occurrence of non-block-diagonal terms in* $\mathbf{K}_0$ *due to the coupling of reversible and irreversible subsystems with a unidirectional energy flow towards the dissipative subsystem. A further contribution of the present paper is the classification of ports in port-GENERIC according to the entropy flow over the port or the entropy creation at the interface.*

2. Preliminaries

2.1. Bond Graphs to Represent Irreversible Phenomena

Bond graphs [12,13] describe the modular composition of multi-physical systems from elementary subsystems (sources, storage, conversion, dissipation if a non-thermodynamic perspective is assumed) via their power interfaces (ports). Basic components for energy routing are 0- and 1-junctions, which generalize Kirchhoff's current and voltage laws. For a short introduction to bond graph modeling, see, e.g., the tutorial papers [21,22].

In this subsection, we introduce *entropy bonds* and point out the elements which are used in both energy and entropy bond graphs to represent irreversible phenomena. The most relevant classical bond graph elements used in this paper are recapitulated in the Appendix A.

2.1.1. Energy and Entropy Bonds

A *bond* of a classical bond graph describes a power interface between two subsystems. It is represented by a half arrow and can be labeled with the two power conjugate variables *effort* e (such as force, voltage or temperature) and *flow* f (such as velocity, current and entropy flow; denoted at the side of the half arrow). The product $P = ef$ denotes the power, and the half arrow indicates the orientation of the energy flow if $ef > 0$, see Figure 1.

In the same manner, we can graphically illustrate *entropy flows* $\sigma = e_\sigma f_\sigma$ related to irreversible phenomena by *entropy bonds*. From $P^{\mathrm{irr}} = \theta\sigma$, a natural (but non-unique) choice of flows and efforts is $e_\sigma = \frac{1}{\theta}$ and $f_\sigma = P^{\mathrm{irr}}$. To distinguish entropy bonds from the classical energy bonds, we set the Greek letter σ next to the half arrow.



Power $P = ef > 0$ Entropy flow $\sigma = e_\sigma f_\sigma > 0$

Figure 1. Entropy and energy bonds. The half arrow points in the direction of a positive energy flow P or positive entropy flow σ, respectively.

2.1.2. Thermal Energy and Entropy Storage

The simplest energy and entropy storage functions, which correspond to the case of constant specific heat, $\bar{U}_d(\theta_d) = c_{v,d}\theta_d$, are

$$\tilde{U}_d(S_d) = c_v e^{S_d/c_{v,d}}, \qquad \hat{S}_d(U_d) = c_v \ln \frac{U_d}{c_{v,d}}. \tag{1}$$

The accents $\bar{(\cdot)}$, $\tilde{(\cdot)}$ and $\hat{(\cdot)}$ denote functions that depend on the temperature, the entropy and the internal energy, respectively; see also Table 1. They can be easily derived by exploiting the second property in Equation (8). See also [4] (Example 2.2), which shows the different expressions for a given free energy function $\bar{\psi}(w, \theta)$. To denote such *non-quadratic* potentials and thus non-linear constitutive equations, the double colon notation is used, see Figure 2. It shows the internal energy and internal entropy storage elements for the thermal energy in the damper considered below.



Figure 2. Energy and entropy storage elements with the co-energy and co-entropy variables derived from the corresponding potential. Here, we show only the one-port elements corresponding to the purely thermal energy storage in the damper, Figures 5 and 6. The multi-port extension is displayed in the bond graphs in Figures 8 and 9 for the Maxwell element.

Table 1. Nomenclature. The accents $\tilde{(\cdot)}$, $\hat{(\cdot)}$ and $\bar{(\cdot)}$ indicate that a function or matrix is expressed in the entropy, internal energy or temperature representation, respectively.

Symbol	Meaning
State variables and representations	
$\boldsymbol{x} \in \mathbb{R}^n$	state vector of a GENERIC system
$\boldsymbol{w} \in \mathbb{R}^{n_w}$	non-thermal (e.g., mechanical) state variables
$\boldsymbol{\tau} \in \mathbb{R}^{n_\tau}$	thermal state variables, $\tau_i \in \{S_i, U_i, \theta_i\}$
$\boldsymbol{S}, \boldsymbol{U}, \boldsymbol{\theta}$	entropies, internal energies and temperatures
$\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$	state in the entropy representation
$\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$	state in the internal energy representation
$\bar{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{\theta})$	state in the temperature representation (not used further)
Energy and entropy functions	
E	total energy $E = \sum_i E_i$
S	total entropy $S = \sum_i S_i$
$\tilde{E}(\boldsymbol{w}, \boldsymbol{S})$	total energy in the entropy representation
$\hat{E}(\boldsymbol{w}, \boldsymbol{U})$	total energy in the internal energy representation
$\tilde{S}(\boldsymbol{w}, \boldsymbol{S})$	total entropy in the entropy representation
$\hat{S}(\boldsymbol{w}, \boldsymbol{U})$	total entropy in the internal energy representation
$\tilde{U}(\boldsymbol{w}, \boldsymbol{S})$	internal energy in the entropy representation
Structure matrices	
$\mathbf{L} = -\mathbf{L}^\top$	Poisson matrix, $\mathbf{L}\nabla S = \mathbf{0}$
$\mathbf{K} = \mathbf{K}^\top \geq 0$	Onsager (friction) matrix, $\mathbf{K}\nabla E = \mathbf{0}$
$\mathbf{M}_E, \mathbf{M}_S$	Jacobians $\partial\tilde{\boldsymbol{x}}/\partial\hat{\boldsymbol{x}}$ and $\partial\hat{\boldsymbol{x}}/\partial\tilde{\boldsymbol{x}}$, $\mathbf{M}_E\mathbf{M}_S = \mathbf{I}$
$\mathbf{L}_0$	canonical Poisson matrix (entropy representation)
$\hat{\mathbf{L}} = \mathbf{M}_S\mathbf{L}_0\mathbf{M}_S^\top$	Poisson matrix in the internal energy representation
$\mathbf{K}_0$	canonical Onsager matrix (internal energy representation)
$\tilde{\mathbf{K}} = \mathbf{M}_E\mathbf{K}_0\mathbf{M}_E^\top$	Onsager matrix in the entropy representation
Ports and interconnection	
$(\boldsymbol{u}^{\text{rev}}, \boldsymbol{y}^{\text{rev}})$, $\mathbf{G}^{\text{rev}}$	reversible ports and input matrix
$(\boldsymbol{u}^{\text{irr}}, \boldsymbol{y}^{\text{irr}})$, $\mathbf{G}^{\text{irr}}$	irreversible ports and input matrix
$(\boldsymbol{u}^{\text{th}}, \boldsymbol{y}^{\text{th}})$, $\mathbf{G}^{\text{th}}$	thermal ports and input matrix

2.1.3. Energy Degradation and Entropy Creation

To represent irreversible phenomena such as the degradation of mechanical power to heat in a damper, the RS (resistive source) element in a classical bond graph describes the

mechanical constitutive relation between force and velocity *and* the conversion of mechanical to thermal power. For the damper considered further below, the corresponding equations are

$$F_d = d_d v_d \qquad \text{and} \qquad F_d v_d = \theta_d \sigma_d^v. \tag{2}$$

The coefficient d_d is the parameter of the RS element, and σ_d^v represents the entropy flow, which is obtained from power continuity a the given temperature θ_d.

To illustrate this entropy creation in the entropy bond graph, a (negative) modulated resistor (MR) is used. Its parameter encodes the relation between the power $d_d v_d^2 = f_\sigma$ (flow variable) and the effort $\frac{1}{\theta_d} = e_\sigma$,

$$f_\sigma = d_d v_d^2 \theta_d e_\sigma. \tag{3}$$

The inverse in the parameter of the MR element in Figure 3 stems from the fact that $d_d v_d^2 \theta_d$ represents formally a conductance, see also Remark 6.



Figure 3. Left: Resistive source (RS) as an energy bond graph element to describe the degradation of mechanical power to heat in a damper. **Right**: The (negative) modulated resistor (MR) describes the associated entropy creation in the entropy bond graph. The cyan arrows indicate that the resistor gain is *modulated* by other bond variables.

2.2. GENERIC and Its Special Form

The *GENERIC* [23] formulation of a closed reversible–irreversible system in *state-space* form, with state vector $\boldsymbol{x} \in \mathbb{R}^n$, reads

$$\dot{\boldsymbol{x}} = \mathbf{L}\nabla E + \mathbf{K}\nabla S, \tag{4}$$

where the skew-symmetric *Poisson* matrix $\mathbf{L} = -\mathbf{L}^\top$ and the symmetric and positive semi-definite *friction* or *Onsager* matrix $\mathbf{K} = \mathbf{K}^\top \geq 0$ satisfy the *non-interaction* conditions

$$\mathbf{L}\nabla S = \mathbf{0}, \qquad \mathbf{K}\nabla E = \mathbf{0}, \tag{5}$$

from which immediately the balance equations

$$\dot{E} = 0, \qquad \dot{S} = \nabla S^\top \mathbf{K} \nabla S \geq 0 \tag{6}$$

for the total energy E and the total entropy S follow. This is in accordance with the first two laws of thermodynamics, i.e., conservation of total energy and non-negative entropy production in a closed system. Oftentimes, the matrices $\mathbf{L}$ and $\mathbf{K}$ depend on the state or co-state variables, see the example in Section 4. For compactness of notation, we omit this dependence here.

The *special* formulation of GENERIC proposed in [4] is based on a splitting of the state vector $\boldsymbol{x} = (\boldsymbol{w}, \boldsymbol{\tau})$ into a part $\boldsymbol{w} = (w_1, \dots, w_{n_w})$, which contains the *non-thermal* (electrical, mechanical, hydraulic etc.) states, and a part $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{n_\tau})$ of thermal state variables. An outstanding feature is the *free choice* of the thermal variable τ_i among the entropy S_i, the internal energy U_i or the temperature θ_i of a thermal subsystem, such that $\tau_i \in \{S_i, U_i, \theta_i\}$. The special GENERIC formulation for finite-dimensional systems without inputs reads

$$\dot{\boldsymbol{x}} = \mathbf{M}_S \mathbf{L}_0 \mathbf{M}_S^\top \nabla E + \mathbf{M}_E \mathbf{K}_0 \mathbf{M}_E^\top \nabla S, \tag{7}$$

where $\mathbf{M}_S$ and $\mathbf{M}_E$ denote the Jacobian matrices pertaining to transformation maps between the different variants of the state vectors, i.e., $\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$, $\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$ and $\bar{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{\theta})$. This notation will be used in the remainder of this work to indicate that certain functions are expressed with respect to one of the possible thermal variables.

2.3. Thermodynamic Transformations

The transformations between the state vectors are derived on the basis of Gibbs' fundamental relation, i.e., the total differential of the internal energy function $U = \tilde{U}(w, S)$ [24]. For simplicity, we first consider the case of scalar variables (we use $\partial_a b$ as a short notation for $\frac{\partial b}{\partial a}$, both for scalar and vector-valued a and b):

$$\mathrm{d}U = \partial_S U \, \mathrm{d}S + \partial_w U \, \mathrm{d}w, \tag{8}$$

where the temperature θ and its inverse are given by

$$\theta = \partial_S U, \qquad \frac{1}{\theta} = \partial_U S. \tag{9}$$

Making use of the previous relations, the internal energy can be written as a function of the temperature,

$$\bar{U}(w, \theta) = \bar{U}(w, \partial_S \tilde{U}(w, S)) = \bar{U}(w, \frac{1}{\partial_U \hat{S}(w, U)}), \tag{10}$$

and a series of relations between the partial derivatives of the thermal variables can be derived, see [4] for more details. Among them, we will exploit in particular

$$\partial_w \tilde{U} = -\theta \partial_w \hat{S}, \qquad \theta = \frac{\partial_\tau U}{\partial_\tau S}, \quad \tau \in \{S, U, \theta\}, \tag{11}$$

see [4] (Equations (2.12) and (2.10)). Note that in the second relation, one assumes that θ, U and S are functions expressed with respect to the same independent thermal variable.

Here, we switch to the system point of view again, considering vectors $\boldsymbol{\tau} \in \mathbb{R}^{n_\tau}$ and $\boldsymbol{w} \in \mathbb{R}^{n_w}$ of thermal and non-thermal state variables. We will restrict our attention to the choices $\boldsymbol{\tau} = \boldsymbol{S}$ and $\boldsymbol{\tau} = \boldsymbol{U}$, as we will use either the internal energies U_i or the entropies S_i as the *thermal storage functions* of the subsystems in the bond graphs for energy and entropy flows. To transform state-space models between $\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$ and $\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$, the relations in (9) and (11) are extended to the vector-valued case, giving the transformations of the time derivatives,

$$\begin{bmatrix} \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{S}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\boldsymbol{\Theta}^{-1} \partial_w \tilde{\boldsymbol{U}} & \boldsymbol{\Theta}^{-1} \end{bmatrix}}_{=\mathbf{M}_E} \begin{bmatrix} \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{U}} \end{bmatrix}, \qquad \begin{bmatrix} \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{U}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\boldsymbol{\Theta} \partial_w \hat{\boldsymbol{S}} & \boldsymbol{\Theta} \end{bmatrix}}_{=\mathbf{M}_S} \begin{bmatrix} \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{S}} \end{bmatrix}, \tag{12}$$

with $\boldsymbol{\Theta} = \mathrm{diag}\{\theta_1, \ldots, \theta_{n_\tau}\}$, and $\partial_w \tilde{\boldsymbol{U}}(\boldsymbol{w}, \boldsymbol{S})$, $\partial_w \hat{\boldsymbol{S}}(\boldsymbol{w}, \boldsymbol{U})$ the corresponding Jacobian submatrices. The matrices

$$\mathbf{M}_E = \frac{\partial \tilde{\boldsymbol{x}}}{\partial \hat{\boldsymbol{x}}} \qquad \text{and} \qquad \mathbf{M}_S = \frac{\partial \hat{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}} \tag{13}$$

represent the Jacobians of the complete thermodynamic transformations, and obviously, $\boldsymbol{M}_E \boldsymbol{M}_S = \boldsymbol{I}$ holds. The gradients of energy and entropy functions transform respectively according to

$$\nabla \hat{E}(\boldsymbol{w}, \boldsymbol{U}) = \mathbf{M}_E^\top \nabla \tilde{E}(\boldsymbol{w}, \boldsymbol{S}), \qquad \nabla \tilde{S}(\boldsymbol{w}, \boldsymbol{S}) = \mathbf{M}_S^\top \nabla \hat{S}(\boldsymbol{w}, \boldsymbol{U}). \tag{14}$$

With $\boldsymbol{\tau} = \boldsymbol{S}$ and $\boldsymbol{\tau} = \boldsymbol{U}$, the two respective special GENERIC formulations are

$$\dot{\tilde{\boldsymbol{x}}} \;=\; \mathbf{L}_0 \nabla \tilde{E}(\boldsymbol{w}, \boldsymbol{S}) \;+\; \tilde{\mathbf{K}} \nabla \tilde{S}(\boldsymbol{w}, \boldsymbol{S}), \tag{15}$$

and

$$\dot{\hat{\boldsymbol{x}}} \;=\; \hat{\mathbf{L}} \nabla \hat{E}(\boldsymbol{w}, \boldsymbol{U}) \;+\; \mathbf{K}_0 \nabla \hat{S}(\boldsymbol{w}, \boldsymbol{U}), \tag{16}$$

where we call $\mathbf{L}_0$ and $\mathbf{K}_0$ the *canonical* Poisson and Onsager matrices for the respective choice of state variables, and where

$$\tilde{\mathbf{K}} = \mathbf{M}_E \mathbf{K}_0 \mathbf{M}_E^\top \quad \text{and} \quad \hat{\mathbf{L}} = \mathbf{M}_S \mathbf{L}_0 \mathbf{M}_S^\top \tag{17}$$

have been introduced as short notation for the other respective matrix. The notation used throughout this paper is summarized in Table 1.

Remark 2. *The general form of the transformation matrices, which includes the temperatures $\tau_i = \theta_i$ as thermodynamic variables, is given in [4] (Section 2.4) and [25]. In this contribution, however, we only make use of $\tau_i = S_i$ and $\tau_i = U_i$, choices that are immediately related to the energy and entropy bond graphs.*

3. A Special GENERIC Form for Open Systems

We propose the extension to *open systems* of the (special) GENERIC formulation, which is guided by the formulation of *port-Hamiltonian* systems. We start with the choice of state variables $\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$, referred to as *entropy formulation*, which allows us to establish the *energy balance* for the open system. The *ports* (u_i, y_i) are the power interfaces to the open system. The duality product of the port variables contributes to the rate of change of the stored total energy $E(\tilde{\boldsymbol{x}})$.

In a second step, we consider the *energy formulation* based on the choice of coordinates $\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$, which reveals the entropy balance. Based on the entropy balance, three classes of ports are distinguished.

3.1. Entropy Formulation—Energy Balance

With the entropies as thermal states, the proposed port-GENERIC formulation reads

$$\dot{\tilde{\boldsymbol{x}}} = \mathbf{L}_0 \nabla \tilde{E} + \tilde{\mathbf{K}} \nabla \tilde{S} + \mathbf{G}^{\text{rev}} \boldsymbol{u}^{\text{rev}} + \mathbf{G}^{\text{irr}} \boldsymbol{u}^{\text{irr}} + \mathbf{G}^{\text{th}} \boldsymbol{u}^{\text{th}} \tag{18a}$$

$$\boldsymbol{y}^{\text{rev}} = (\mathbf{G}^{\text{rev}})^\top \nabla \tilde{E} \tag{18b}$$

$$\boldsymbol{y}^{\text{irr}} = (\mathbf{G}^{\text{irr}})^\top \nabla \tilde{E} \tag{18c}$$

$$\boldsymbol{y}^{\text{th}} = (\mathbf{G}^{\text{th}})^\top \nabla \tilde{E}. \tag{18d}$$

For compactness of notation, we write all matrices (except for the Onsager matrix) without an accent, and we omit all possible dependencies on state and co-state variables. $\nabla \tilde{E}$ and $\nabla \tilde{S}$ are the short notations for the gradients $\nabla E(\tilde{\boldsymbol{x}})$ and $\nabla S(\tilde{\boldsymbol{x}})$ of the total energy and entropy. The formulation features three categories of ports:

- Ports with potentially bidirectional, *reversible* energy flow $(u_i^{\text{rev}}, y_i^{\text{rev}})$. *Potentially* refers to the fact that closing such a port with a dissipative port of another system leads to a unidirectional energy flow, as will be observed in the coupling of the mass and the damper in the didactic example of the next section.
- Ports $(u_i^{\text{irr}}, y_i^{\text{irr}})$, which are immediately related to *irreversible* energy conversion, with a unidirectional energy/entropy flow.
- Ports $(u_i^{\text{th}}, y_i^{\text{th}})$, which carry thermal energy (heat). Such ports are related to *thermal interfaces*, at which bidirectional energy/entropy flow occurs. At a thermal interface, entropy creation can occur, a typical case being heat conduction described through Fourier's law.

The energy balance with the three categories of ports reads (employing the non-interaction condition $\tilde{\mathbf{K}} \nabla \tilde{E} = \mathbf{0}$)

$$\dot{E}(\tilde{\boldsymbol{x}}) = (\boldsymbol{y}^{\text{rev}})^\top \boldsymbol{u}^{\text{rev}} + \underbrace{(\boldsymbol{y}^{\text{irr}})^\top \boldsymbol{u}^{\text{irr}}}_{\geq 0} + (\boldsymbol{y}^{\text{th}})^\top \boldsymbol{u}^{\text{th}}. \tag{19}$$

The inequality below the second term on the right indicates the immediate degradation of this supplied power to heat, increasing the internal energy part of E.

Remark 3. *Just as for* port-Hamiltonian *systems, extensions of the above Formulation* (18) *are possible, e.g., with direct feedthrough in the output equations, see, e.g., [26]. We refrain from this generalization here to concentrate on the physical examples and their bond graphs below.*

3.2. Energy Formulation—Entropy Balance

With the choice of state variables $\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$, and exploiting the Jacobians of the state transformations in (12), the port-GENERIC differential equation reads

$$\dot{\hat{\boldsymbol{x}}} = \hat{\mathbf{L}}\nabla\hat{E} + \mathbf{K}_0\nabla\hat{S} + \mathbf{M}_S\mathbf{G}^{\mathrm{rev}}\boldsymbol{u}^{\mathrm{rev}} + \mathbf{M}_S\mathbf{G}^{\mathrm{irr}}\boldsymbol{u}^{\mathrm{irr}} + \mathbf{M}_S\boldsymbol{G}^{\mathrm{th}}\boldsymbol{u}^{\mathrm{th}}. \tag{20}$$

As the outputs, which are defined in (18b–d) in a power-conjugate way, do not directly enter the entropy balance equation, we omit them here. Exploiting the non-interaction condition $\hat{\mathbf{L}}\nabla\hat{S} = \mathbf{0}$ and the transformation of the entropy gradient $\mathbf{M}_S^\top\nabla\hat{S} = \nabla\tilde{S}$ according to (14), the entropy balance can be written

$$\dot{S} = \underbrace{(\nabla\hat{S})^\top\mathbf{K}_0\nabla\hat{S}}_{\geq 0} + \underbrace{(\nabla\tilde{S})^\top\mathbf{G}^{\mathrm{rev}}\boldsymbol{u}^{\mathrm{rev}}}_{=0} + \underbrace{(\nabla\tilde{S})^\top\mathbf{G}^{\mathrm{irr}}\boldsymbol{u}^{\mathrm{irr}}}_{\geq 0} + \underbrace{(\nabla\tilde{S})^\top\mathbf{G}^{\mathrm{th}}\boldsymbol{u}^{\mathrm{th}}}_{\text{indefinite}}. \tag{21}$$

With the simple structure of the entropy gradient $\nabla\tilde{S}^\top = (\mathbf{0}^\top, \mathbf{1}^\top)$, we can characterize different types of ports in terms of entropy creation:

- The *reversible* ports $(\boldsymbol{u}^{\mathrm{rev}}, \boldsymbol{y}^{\mathrm{rev}})$ are characterized by the property

$$(\nabla\tilde{S})^\top\mathbf{G}^{\mathrm{rev}} = \mathbf{0}^\top \tag{22}$$

  of the corresponding input matrix. The corresponding inputs induce only dynamics of the non-thermal state variables $\boldsymbol{w}$.
- *Irreversible* ports $(\boldsymbol{u}^{\mathrm{irr}}, \boldsymbol{y}^{\mathrm{irr}})$ feature non-negative entropy creation

$$(\nabla\tilde{S})^\top\boldsymbol{G}^{\mathrm{irr}}\boldsymbol{u}^{\mathrm{irr}} \geq 0 \qquad \text{for} \qquad \boldsymbol{u}^{\mathrm{irr}} \neq \mathbf{0}. \tag{23}$$

- At *thermal* ports $(\boldsymbol{u}^{\mathrm{th}}, \boldsymbol{y}^{\mathrm{th}})$, the entropy flow rate $(\nabla\tilde{S})^\top\boldsymbol{G}^{\mathrm{th}}\boldsymbol{u}^{\mathrm{th}}$ is *indefinite* for $\boldsymbol{u}^{\mathrm{th}} \neq \mathbf{0}$.

Remark 4. *It is straightforward to verify that* (18) *with combined in- and outputs* $\boldsymbol{u} = (\boldsymbol{u}^{\mathrm{rev}}, \boldsymbol{u}^{\mathrm{irr}}, \boldsymbol{u}^{\mathrm{th}})$ *and* $\boldsymbol{y} = (\boldsymbol{y}^{\mathrm{rev}}, \boldsymbol{y}^{\mathrm{irr}}, \boldsymbol{y}^{\mathrm{th}})$ *corresponds to a port-Hamiltonian system with dissipation [10,11] in canonical form*

$$\dot{\boldsymbol{x}} = (\mathbf{J} - \mathbf{R})\nabla H(\boldsymbol{x}) + \mathbf{G}\boldsymbol{u} \tag{24a}$$

$$\boldsymbol{y} = \mathbf{G}^\top\nabla H(\boldsymbol{x}) \tag{24b}$$

with interconnection matrix $\mathbf{J} = \mathbf{L}$ *and dissipation matrix* $\mathbf{R} = \frac{1}{T_0}\mathbf{K}$*, where* T_0 *is a constant temperature, if the* exergy $H = E - T_0S$ *is taken as a single potential—predominantly termed* Hamiltonian *in that context. See also Remark 2.1 in [4].*

4. Didactic Example: Lumped Thermo-Visco-Elastic System

The purpose of the model problem, as sketched in Figure 4, is to show the relation of port-GENERIC subsystems to their energy and entropy bond graphs, and to demonstrate how an interconnected model results from the composition of subsystems via mechanical and thermal coupling conditions.

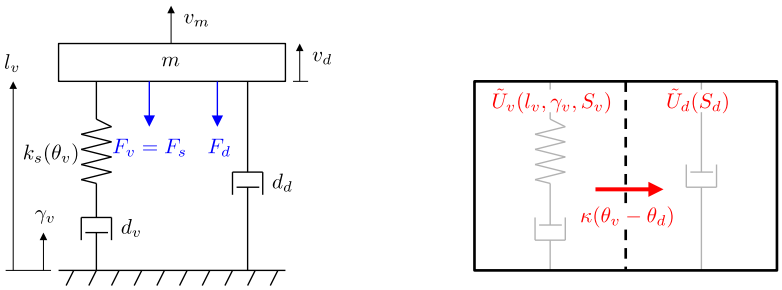

Figure 4. Sketch of the lumped thermo-visco-elastic model problem: The thermo-mechanical behavior is modeled by the parallel interconnection of a pure damper (subscript d) with a visco-elastic Maxwell element (subscript v). The corresponding forces F_d and F_v act on a lumped mass m. Both damper and Maxwell element constitute thermodynamic subsystems, which exchange heat according to Fourier's law (lumped thermal conductance κ). For simplicity, we consider the overall system as isolated, closed by an adiabatic wall.

A damper (subscript d) and a visco-elastic Maxwell element (subscript v) are connected to a mass m as a purely mechanical subsystem (Figure 4, left). The damper and the Maxwell element are moreover considered as two thermal subsystems, which exchange heat according to Fourier's law (Figure 4, right). For simplicity, the system is thermally closed by adiabatic walls. The two subsystems heat up due to the conversion of mechanical energy to heat in the dampers. The forces in the dampers are given by the linear constitutive equations

$$F_d = d_d v_d, \qquad F_v = d_v \dot{\gamma}_v, \tag{25}$$

where v_d is the velocity of the simple damper element, γ_v is the internal displacement variable associated with the damper in the Maxwell element, and d_d, d_v are the respective damping coefficients. The spring in the Maxwell element (subscript s and temperature-dependent stiffness k_s) is modeled by the force-displacement relation

$$F_s = k_s(\theta_v)(l_v - \gamma_v), \tag{26}$$

which can be derived from an internal energy function $\tilde{U}_v(l_v, \gamma_v, S_v)$ or a Helmholtz free energy $\bar{\Psi}_v(l_v, \gamma_v, \theta_v)$ according to

$$F_s = \partial_{l_v} \tilde{U}_v = \partial_{l_v} \bar{\Psi}_v. \tag{27}$$

For later use, we note that

$$\tilde{U}_v(l_v, \gamma_v, S_v) = \tilde{U}_s(l_v - \gamma_v, S_v), \tag{28}$$

where $\tilde{U}_s$ denotes the internal energy of the Maxwell element as a function of the spring elongation $l_v - \gamma_v$ and the entropy S_v, in accordance with (26). The last relation implies $\partial_{l_v} \tilde{U}_v = -\partial_{\gamma_v} \tilde{U}_v$. Finally, note that the series connection of spring and damper in the Maxwell element implies $F_s = F_v$.

In what follows, we start from the subsystem Formulation (18) in state variables $\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$ with *reversible*, *irreversible* and *thermal* ports. The matrices of the port-GENERIC formulation are defined below for each subsystem. We write out the gradients of the subsystem energies $\nabla \tilde{E}_i$ and entropies $\nabla \tilde{S}_i$ to underscore the respective roles of the physical variables.

4.1. Port-GENERIC Damper Model

The damper, which is modeled as a thermodynamic subsystem converting mechanical power into heat, is characterized by its internal energy $\tilde{U}_d(S_d)$ in terms of the entropy S_d,

from which the temperature $\theta_d = \frac{\partial \tilde{U}_d}{\partial S_d}$ is derived. Moreover, $\frac{\partial \tilde{S}_d}{\partial S_d} = 1$. Its port-GENERIC Formulation (18) reads

$$\dot{S}_d = 0 \cdot \theta_d + 0 \cdot 1 + \frac{d_d v_d}{\theta_d} \cdot v_d + 1 \cdot \sigma_d^{\text{in}} \tag{29a}$$

$$F_d = \frac{d_d v_d}{\theta_d} \cdot \theta_d \tag{29b}$$

$$\theta_d = 1 \cdot \theta_d \tag{29c}$$

with the irreversible port (v_d, F_d) and the thermal port $(\sigma_d^{\text{in}}, \theta_d)$. The mechanical output Equation (29b) describes the linear constitutive law of the damper according to (25), and the ordinary differential Equation (29a) represents the increase in entropy due to the mechanical power loss $d_d v_d^2$ and the entropy flow σ_d^{in} caused by external heat supply.

While both elements L_d^0 and $\tilde{K}_d$ of the port-GENERIC representation are zero, the input gains $G_d^{\text{irr}} = \frac{d_d v_d}{\theta_d}$ and $G_d^{\text{th}} = 1$ appear accordingly in the definitions of the power-conjugate outputs (29b,c). The port variables emerge in the energy balance equation

$$\dot{\tilde{U}}_d(S_d) = \frac{\partial U_d}{\partial S_d} \dot{S}_d = F_d v_d + \theta_d \sigma_d^{\text{in}}, \tag{30}$$

see the energy bond graph in Figure 5.

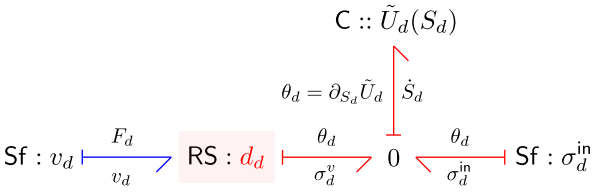

Figure 5. Energy bond graph for the damper. The blue bond represents purely mechanical power, while the red bonds are related to heat transfer and entropy flow. Only their entropy flow counterparts appear in the entropy bond graph. The bonds at the flow source elements (Sf) carry the dual power variables of the irreversible and the thermal port, respectively.

The entropy creation σ_d^v due to the imposed velocity v_d can be expressed as a quadratic form in $\nabla \hat{S}_d(U_d) = \frac{1}{\theta_d}$:

$$\sigma_d^v = d_d v_d^2 \theta_d \left(\frac{1}{\theta_d}\right)^2 \geq 0. \tag{31}$$

The non-negativity confirms that (v_d, F_d) is an irreversible port. The factor $d_d v_d^2 \theta_d$ is exactly the inverse resistance (i.e., the conductance) of the MR element in the entropy bond graph depicted in Figure 6. The outgoing orientation of the attached bond indicates that the MR element represents entropy creation. (Recall that in classical bond graphs, (modulated) resistive elements represent dissipation, i.e., degradation of energy.)

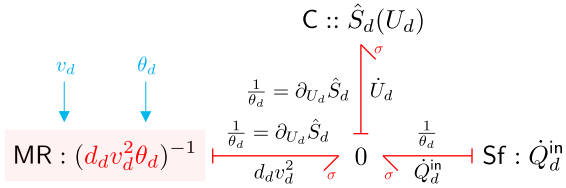

Figure 6. Entropy bond graph for the damper. The symbol σ at the tips indicates that the bonds carry an entropy flow, as the product of entropy-flow-conjugate bond variables. The conductance $d_d v_d^2 \theta_d$ is *modulated* by both the temperature (co-state variable) and the velocity (input variable).

Remark 5. *The presented construction of energy and entropy bond graphs for constant linear resistive relations such as $F_d = d_d v_d$ can be directly extended to nonlinear relations of the type*

$F_d = d_d(v_d)v_d$ via modulated RS elements in the energy bond graphs. For nonlinear relations given in the form $F_d = f_d(v_d)$, one could borrow the notation from nonlinear energy storage elements using the Rayleigh dissipation potential.

Remark 6. *The inverse in the MR element of Figure 6 suggests a potential realization issue at velocity $v_d = 0$. However, note that the causal stroke indicates the causality $f_\sigma = R_\sigma^{-1} e_\sigma$ with $R_\sigma = (d_d v_d^2 \theta_d)^{-1}$ and f_σ and e_σ the dissipated power $d_d v_d^2$ and the inverse temperature $\frac{1}{\theta_d}$ as flows and efforts in the entropy bond graph. Consequently, by the double inverse, no singularity and realization issue occurs at $v_d = 0$.*

4.2. Port-GENERIC Model of the Mass

The mass is a purely mechanical, non-thermodynamic subsystem, whose energy coincides with the kinetic energy, which can be written in terms of the momentum p_m as

$$T(p_m) = \frac{1}{2m} p_m^2, \qquad v_m = \frac{\partial T(p_m)}{\partial p_m} = \frac{p_m}{m}. \tag{32}$$

Here, v_m is the velocity of the mass. The port-GENERIC representation (18) reduces to the state-space model

$$\dot{p}_m = 0 \cdot v_m + 1 \cdot F_m, \tag{33a}$$
$$v_m = 1 \cdot v_m. \tag{33b}$$

It does not feature a thermal port, and only the reversible mechanical port (F_m, v_m) drives the energy balance

$$\dot{T}(p_m) = F_m v_m. \tag{34}$$

Therefore, (33) is equivalent to a *port-Hamiltonian* model, see also Figure 7 for its (energy) bond graph.

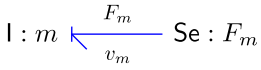

Figure 7. Bond graph for the mass: The effort source imposes the force F_m on the mass. In the (linear) I-type storage element, the effort (here the force) is integrated to give the momentum according to $\dot{p}_m = F_m$; the flow (here the velocity) is obtained by division by the parameter m: $v_m = \frac{p_m}{m}$.

4.3. Interconnection of Mass and Damper

The open-loop mass and damper models dealt with in the previous two subsections can now be combined into a single port-GENERIC state representation of the form (18) with state vector $\tilde{\boldsymbol{x}}_{md} = (p_m, S_d)$. The total energy and entropy are then given by

$$\tilde{E}_{md}(p_m, S_d) = \frac{1}{2m} p_m^2 + \tilde{U}_d(S_d) \qquad \text{and} \qquad \tilde{S}_{md}(p_m, S_d) = S_d. \tag{35}$$

Putting together (33) and (29), the combined mass-damper system can be written as

$$\dot{\tilde{\boldsymbol{x}}}_{md} = \mathbf{0} \cdot \nabla \tilde{E}_{md} + \mathbf{0} \cdot \nabla \tilde{S}_{md} + \mathbf{G}_m^{\text{rev}} F_m + \mathbf{G}_d^{\text{irr}} v_d + \mathbf{G}_d^{\text{th}} \sigma_d^{\text{in}} \tag{36a}$$
$$v_m = (\mathbf{G}_m^{\text{rev}})^\top \nabla \tilde{E}_{md} \tag{36b}$$
$$F_d = (\mathbf{G}_d^{\text{irr}})^\top \nabla \tilde{E}_{md} \tag{36c}$$
$$\theta_d = (\mathbf{G}_d^{\text{th}})^\top \nabla \tilde{E}_{md}, \tag{36d}$$

where

$$\nabla \tilde{E}_{md} = \begin{bmatrix} v_m \\ \theta_d \end{bmatrix}, \qquad \nabla \tilde{S}_{md} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{37}$$

and the input vectors

$$\mathbf{G}_m^{\text{rev}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{G}_d^{\text{irr}} = \begin{bmatrix} 0 \\ \frac{d_d v_d}{\theta_d} \end{bmatrix}, \qquad \mathbf{G}_d^{\text{th}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{38}$$

with the superscripts referring to the categories of the corresponding ports (recall that $v_d F_d = d_d v_d^2 \geq 0$ at the irreversible port). Closing the thermal port via Fourier's law of heat conduction and the description of the resulting thermal interface are discussed in Section 4.5.2.

We now consider the purely mechanical interconnection of the mass and the damper. To this end, we impose the kinematic condition $v_d = v_m = v_{md}$ along with the law of interaction $F_m = -F_d$. In addition to that, we provide the new port (F_{md}, v_{md}) belonging to the interconnected mass-damper system. The power-preserving interconnection conditions are summarized as

$$\begin{bmatrix} F_m \\ v_d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_d \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{md} \tag{39a}$$

$$v_{md} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_d \end{bmatrix}. \tag{39b}$$

We now aim for the port-GENERIC formulation of the combined mass-damper system by using the interconnection condition (39a) to close the ports (F_m, v_m) and (v_d, F_d) and to eliminate the corresponding in- and outputs from (36). A straightforward calculation yields

$$\dot{p}_m = -d_d v_m + F_{md} \tag{40a}$$

$$\dot{S}_d = \frac{d_d v_m^2}{\theta_d} + \sigma_d^{\text{in}}, \tag{40b}$$

where (36c), i.e., $F_d = d_d v_d$, has been used. The above equations can again be written in port-GENERIC form (18), leading to

$$\begin{bmatrix} \dot{p}_m \\ \dot{S}_d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ \theta_d \end{bmatrix} + \begin{bmatrix} d_d \theta_d & -d_d v_m \\ -d_d v_m & \frac{d_d v_m^2}{\theta_d} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{md} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma_d^{\text{in}} \tag{41a}$$

$$v_{md} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ \theta_d \end{bmatrix} \tag{41b}$$

$$\theta_d = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_m \\ \theta_d \end{bmatrix}. \tag{41c}$$

It can easily be verified that (41a) is equivalent to (40). The Onsager matrix $\tilde{\mathbf{K}}_{md}$ in (41a) has been obtained by adhering to the symmetry condition $\tilde{\mathbf{K}}_{md} = \tilde{\mathbf{K}}_{md}^\top$ and the non-interaction condition $\tilde{\mathbf{K}}_{md} \nabla \tilde{E} = \mathbf{0}$.

Remark 7. *Following the port-Hamiltonian formalism in a straightforward way, the interconnection conditions* (39a) *applied to* (36a) *generate the right-hand-side term*

$$\mathbf{G}^{\text{ic}} \mathbf{J}^{\text{ic}} (\mathbf{G}^{\text{ic}})^\top \nabla \tilde{E} = \begin{bmatrix} 0 & -\frac{d_d v_m}{\theta_d} \\ \frac{d_d v_m}{\theta_d} & 0 \end{bmatrix} \begin{bmatrix} v_m \\ \theta_d \end{bmatrix} = \begin{bmatrix} -d_d v_m \\ d_d v_m^2 \end{bmatrix}, \tag{42}$$

with the matrices

$$\mathbf{J}^{\text{ic}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{G}^{\text{ic}} = \begin{bmatrix} \mathbf{G}_d^{\text{irr}} & \mathbf{G}_m^{\text{rev}} \end{bmatrix}. \tag{43}$$

Skew-symmetry of the matrix in (42) indicates the power-preserving character of the interconnection. The modulation *with the port variables v_m and θ_d is a typical feature of irreversible couplings, see, e.g., [15]. However, for the port-GENERIC model, closing the irreversible port (v_d, F_d) must result in a modified dissipative structure. Therefore, $\tilde{\mathbf{K}}_{md} = \tilde{\mathbf{K}}_{md}^{\top}$ stems from the matching*

$$\mathbf{G}^{\mathrm{ic}}\mathbf{J}^{\mathrm{ic}}(\mathbf{G}^{\mathrm{ic}})^{\top}\nabla\tilde{E} = \tilde{\mathbf{K}}_{md}\nabla\tilde{S} \tag{44}$$

with $\nabla\tilde{S} = (0, 1)$, and the non-interaction condition $\tilde{\mathbf{K}}_{md}\nabla\tilde{E}_{md} = \mathbf{0}$.

4.4. Port-GENERIC Model of the Visco-Elastic Subsystem

As described above, the state vector for the visco-elastic subsystem (subscript v) is composed of the mechanical configuration variable l_v (total length), the internal variable γ_v of the series damper, whose time derivative is directly related to entropy creation, and the corresponding entropy variable S_v. In particular, the total energy and entropy of the Maxwell element read

$$\tilde{E}_v = \tilde{U}_v(l_v, \gamma_v, S_v) \qquad \text{and} \qquad \tilde{S}_v = S_v, \tag{45}$$

where $\tilde{U}_v$ is the internal energy introduced in (28). Now, we obtain

$$\nabla\tilde{E}_v = \begin{bmatrix} F_v \\ -F_v \\ \theta_v \end{bmatrix} \qquad \text{and} \qquad \nabla\tilde{S}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{46}$$

The evolution equations pertaining to the Maxwell element

$$\dot{l}_v = v_v \tag{47a}$$

$$\dot{\gamma}_v = \frac{F_v}{d_v} = -\frac{1}{d_v}\partial_{\gamma_v}\tilde{U}_v \tag{47b}$$

$$\dot{S}_v = \frac{F_v\dot{\gamma}_v}{\theta_v} + \sigma_v^{\mathrm{in}} \tag{47c}$$

can now be brought into the port-GENERIC form (18). With the velocity v_v and the entropy flow rate σ_v^{in} of the external heat supply as inputs, we obtain

$$\begin{bmatrix} \dot{l}_v \\ \dot{\gamma}_v \\ \dot{S}_v \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} F_v \\ -F_v \\ \theta_v \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\theta_v}{d_v} & \frac{F_v}{d_v} \\ 0 & \frac{F_v}{d_v} & \frac{F_v^2}{d_v\theta_v} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\sigma_v^{\mathrm{in}} \tag{48a}$$

$$F_v = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} F_v \\ -F_v \\ \theta_v \end{bmatrix} \tag{48b}$$

$$\theta_v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} F_v \\ -F_v \\ \theta_v \end{bmatrix}. \tag{48c}$$

The Onsager matrix $\tilde{\mathbf{K}}_v$ can be read directly from (48a). Note that the third column of $\tilde{\mathbf{K}}_v$ is immediately derived from (47b,c). The second column follows from the symmetry of $\tilde{\mathbf{K}}_v$ and the non-interaction condition $\tilde{\mathbf{K}}_v\nabla\tilde{E}_v = \mathbf{0}$. In contrast to the pure damper treated in Section 4.1, the Onsager matrix $\tilde{\mathbf{K}}_v$ here is non-zero: While entropy production in the former case is strictly related to a non-zero input, here it depends on the (internal) co-state variable $F_v = \partial_{l_v}\tilde{U}_v = -\partial_{\gamma_v}\tilde{U}_v$. The energy and entropy bond graphs for the visco-elastic subsystem are shown in Figures 8 and 9.

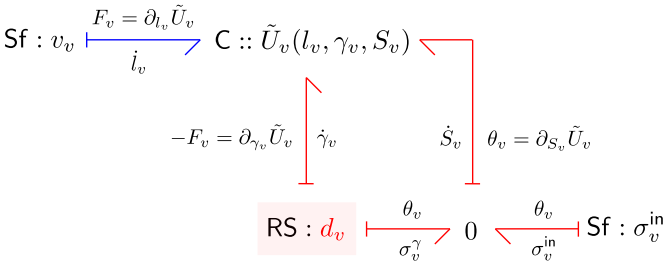

Figure 8. Energy bond graph for the Maxwell element. Other than in the damper example, the mechanical power on the blue bond is not immediately converted into heat, but via the internal velocity $\dot{\gamma}_v$.

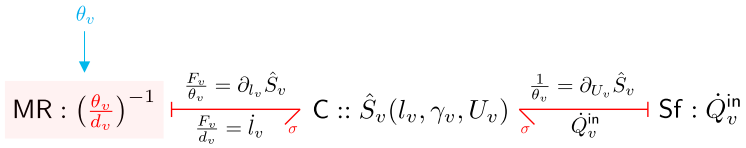

Figure 9. Entropy bond graph for the Maxwell element. The internal entropy generation rate σ_v^γ can be represented as $\sigma_v^\gamma = \frac{F_v^2}{d_v\theta_v} = \frac{\theta_v}{d_v}(\partial_{l_v}\hat{S}_v)^2$, i.e., via the inverse parameter of the MR element.

Finally, we remark that the Onsager matrix takes a particularly simple (canonical) form in energy coordinates $\hat{\boldsymbol{x}} = (l_v, \gamma_v, U_v)$, leading to

$$\mathbf{K}_{v,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\theta_v}{d_v} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

This matrix can be deduced from $\mathbf{K}_{v,0} = \mathbf{M}_E^{-1}\tilde{\mathbf{K}}_v\mathbf{M}_E^{-\top}$ with the inverse Jacobian

$$\mathbf{M}_E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_{l_v}\tilde{U}_v & \partial_{\gamma_v}\tilde{U}_v & \theta_v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_v & -F_v & \theta_v \end{bmatrix}. \tag{50}$$

The only non-zero diagonal element of $\mathbf{K}_{v,0}$ is the conductance of the MR element depicted in the entropy bond graph, see Figure 9.

4.5. Interconnection of the Mass-Damper System with the Maxwell Element

Finally, we couple the mass-damper system with the Maxwell element. The interconnection of the two associated port-GENERIC models yields a joint port-GENERIC model of the complete thermo-visco-elastic system depicted in Figure 4. The coupling of both systems leads to the state vector $\tilde{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{S})$, with $\boldsymbol{w} = (p_m, l_v, \gamma_v)$ and $\boldsymbol{S} = (S_d, S_v)$. The total energy and entropy of the joint system are given by

$$\tilde{E} = \frac{1}{2m}p_m^2 + \tilde{U}_d(S_d) + \tilde{U}_v(l_v, \gamma_v, S_v) \qquad \text{and} \qquad \tilde{S} = S_d + S_v, \tag{51}$$

leading to the gradients

$$\nabla\tilde{E} = \begin{bmatrix} v_m \\ F_v \\ -F_v \\ \theta_d \\ \theta_v \end{bmatrix} \qquad \text{and} \qquad \nabla\tilde{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{52}$$

4.5.1. Mechanical Interconnection

Interconnecting the models of the mass-damper system (41) and the Maxwell element (48) on a purely mechanical level via

$$\begin{bmatrix} F_{md} \\ v_v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_v \end{bmatrix} \tag{53}$$

gives

$$\begin{bmatrix} \dot{p}_m \\ \dot{l}_v \\ \dot{\gamma}_v \\ \dot{S}_d \\ \dot{S}_v \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_v \\ -F_v \\ \theta_d \\ \theta_v \end{bmatrix} + \begin{bmatrix} d_d\theta_d & 0 & 0 & -d_d v_m & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta_v}{d_v} & 0 & \frac{F_v}{d_v} \\ -d_d v_m & 0 & 0 & \frac{d_d v_m^2}{\theta_d} & 0 \\ 0 & 0 & \frac{F_v}{d_v} & 0 & \frac{F_v^2}{d_v\theta_v} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_d^{\text{in}} \\ \sigma_v^{\text{in}} \end{bmatrix} \tag{54a}$$

$$\begin{bmatrix} \theta_d \\ \theta_v \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_m \\ F_v \\ -F_v \\ \theta_d \\ \theta_v \end{bmatrix}. \tag{54b}$$

The Poisson matrix $\mathbf{L}_0$ of the interconnected system results (see also the first step of Remark 7) from the skew-symmetry of the interconnection matrix $\mathbf{J}^{\text{ic}}$ in the coupling conditions (53) and the power-conjugate definition of in- and outputs: $\mathbf{L}_0 = \mathbf{G}^{\text{ic}}\mathbf{J}^{\text{ic}}(\mathbf{G}^{\text{ic}})^\top = -\mathbf{L}_0^\top$. Here, $\mathbf{G}^{\text{ic}} = (\boldsymbol{e}_1, \boldsymbol{e}_2) \in \mathbb{R}^{2\times 5}$ is composed of the first two unit vectors, representing the input vectors for F_{md} and v_v, respectively. The ports (F_{md}, v_{md}) and (v_v, F_v) are classified as reversible, as they are not related to immediate degradation of energy to heat. Accordingly, it is obvious that the non-interaction condition $\mathbf{L}_0\nabla\tilde{S} = \mathbf{0}$ is satisfied.

4.5.2. Thermal Interface

For simplicity, we consider the interconnected system closed, i.e., with rigid and adiabatic walls, which do not allow for the exchange of work and heat, cf. Figure 4. Heat exchange occurs only internally, between the two thermal subsystems (mass-)damper and Maxwell element. In particular, heat exchange is modeled by Fourier's law, leading to the heat flow

$$\dot{Q}_d^{\text{in}} = \kappa(\theta_v - \theta_d) = -\dot{Q}_v^{\text{in}}, \tag{55}$$

where $\dot{Q}_d^{\text{in}}$ and $\dot{Q}_v^{\text{in}}$ are heat flows into the respective subsystems, and $\kappa > 0$ is a lumped heat conductivity coefficient. Now, the entropy flows to the damper and the Maxwell element due to heat conduction are given by $\sigma_d^{\text{in}} = \frac{\dot{Q}_d^{\text{in}}}{\theta_d}$ and $\sigma_v^{\text{in}} = \frac{\dot{Q}_v^{\text{in}}}{\theta_v}$, respectively. Accordingly, the thermal interface conditions can be written as

$$\begin{bmatrix} \sigma_d^{\text{in}} \\ \sigma_v^{\text{in}} \end{bmatrix} = \kappa \begin{bmatrix} \frac{\theta_v}{\theta_d} & -1 \\ -1 & \frac{\theta_d}{\theta_v} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{56}$$

which can be directly inserted into (54), yielding the GENERIC state differential equations of the resulting closed system:

$$\begin{bmatrix} \dot{p}_m \\ \dot{l}_v \\ \dot{\gamma}_v \\ \dot{S}_d \\ \dot{S}_v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{L}_0} \begin{bmatrix} v_m \\ F_v \\ -F_v \\ \theta_d \\ \theta_v \end{bmatrix} + \underbrace{\begin{bmatrix} d_d\theta_d & 0 & 0 & -d_d v_m & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta_v}{d_v} & 0 & \frac{F_v}{d_v} \\ -d_d v_m & 0 & 0 & \frac{d_d v_m^2}{\theta_d} + \kappa\frac{\theta_v}{\theta_d} & -\kappa \\ 0 & 0 & \frac{F_v}{d_v} & -\kappa & \frac{F_v^2}{d_v\theta_v} + \kappa\frac{\theta_d}{\theta_v} \end{bmatrix}}_{\tilde{\mathbf{K}}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \tag{57}$$

Note that the positioning of the entries due to heat conduction in the Onsager matrix $\tilde{\mathbf{K}}$ (and, correspondingly, in (56)) is governed by the symmetry condition $\tilde{\mathbf{K}}^\top = \tilde{\mathbf{K}}$ and the non-interaction condition $\tilde{\mathbf{K}}\nabla\tilde{E} = \mathbf{0}$.

Next, we point out the effect of the thermal interface on the GENERIC representation in terms of energy variables. Choosing $\tau_i = U_i$, the interface conditions (56) can be rewritten (cf. (17), with $\mathbf{M}_E^{-1} = \operatorname{diag}\{\theta_d, \theta_v\}$) as

$$\begin{bmatrix} \dot{Q}_d^{\text{in}} \\ \dot{Q}_v^{\text{in}} \end{bmatrix} = \kappa\theta_d\theta_v \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\theta_d} \\ \frac{1}{\theta_v} \end{bmatrix}. \tag{58}$$

While $\kappa\theta_d\theta_v$ is again the conductance of the MR element in the entropy bond graph representing the thermal interface, see Figure 10, the rank-deficient matrix stems from the difference of the reciprocal temperatures, which drives the entropy production originating in the heat transfer:

$$\Delta\sigma = \kappa\theta_d\theta_v \left(\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\theta_d} \\ \frac{1}{\theta_v} \end{bmatrix} \right)^2. \tag{59}$$

This difference is reflected by the topology of the entropy bond graph in Figure 10 through the 1-junction.

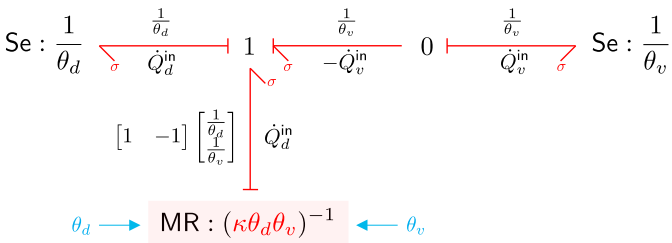

Figure 10. Entropy bond graph for the thermal interface.

To conclude this section, the overall GENERIC model for the interconnected system in energy coordinates $\hat{\boldsymbol{x}} = (\boldsymbol{w}, \boldsymbol{U})$ with the vector of internal energies $\boldsymbol{U} = (U_d, U_v)$ is

$$\begin{bmatrix} \dot{p}_m \\ \dot{l}_v \\ \dot{\gamma}_v \\ \dot{U}_d \\ \dot{U}_v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & -F_v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ F_v & 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{\mathbf{L}}} \begin{bmatrix} v_m \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} d_d\theta_d & 0 & 0 & -d_d\theta_d v_m & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta_v}{d_v} & 0 & 0 \\ -d_d\theta_d v_m & 0 & 0 & d_d\theta_d v_m^2 + \kappa\theta_d\theta_v & -\kappa\theta_d\theta_v \\ 0 & 0 & 0 & -\kappa\,\theta_d\,\theta_v & \kappa\,\theta_d\,\theta_v \end{bmatrix}}_{\mathbf{K}_0} \begin{bmatrix} 0 \\ -\frac{F_v}{\theta_v} \\ \frac{F_v}{\theta_v} \\ \frac{1}{\theta_d} \\ \frac{1}{\theta_v} \end{bmatrix}. \tag{60}$$

While the Poisson matrix $\hat{\mathbf{L}}$ features an additional coupling (compared to the canonical matrix $\mathbf{L}_0$, which depicts the purely reversible energy exchange between subsystems), modulated by the spring force of the Maxwell element, all conductances of the MR elements in the entropy bond graphs appear on the (block) diagonal of the canonical Onsager matrix $\mathbf{K}_0$. The off-diagonal elements hint at the unidirectional (dissipative/irreversible) flow of mechanical energy in the damper. The matrices in (60) have all the features that a GENERIC model needs, see Section 2.2. While symmetry and skew-symmetry, as well as the non-interaction conditions can be checked directly, positive semi-definiteness of $\mathbf{K}_0$ follows from the fact that it is the sum of three symmetric rank-one terms $\mathbf{K}_0 = d_d\theta_d\,\boldsymbol{a}\boldsymbol{a}^\top + \frac{\theta_v}{d_v}\boldsymbol{e}_3\boldsymbol{e}_3^\top + \kappa\theta_d\theta_v\,\boldsymbol{b}\boldsymbol{b}^\top$ with $\boldsymbol{a} = \boldsymbol{e}_1 - v_m\boldsymbol{e}_4$ and $\boldsymbol{b} = \boldsymbol{e}_4 - \boldsymbol{e}_5$, where $\boldsymbol{e}_i$ denotes the i-th unit vector. Eventually, the conservation of energy and the non-negative entropy production are ensured by design.

This didactic section aimed to clarify the thermal and non-thermal ports in finite-dimensional port-GENERIC subsystems, their interconnection with unidirectional or bidirectional energy flow (related to entropy generation or not), and also the thermal coupling via a heat-conducting interface. The corresponding modifications of the Poisson and Onsager matrices were highlighted, and related to the energy and entropy bond graphs of the subsystems.

5. Conclusions

We have proposed a port-based formulation of Mielke's special GENERIC framework for *open* systems, which, in analogy to port-Hamiltonian systems, allows the modular composition of models via power-conjugate ports.

We have shown the systematic derivation of the canonical Poisson and Onsager matrices for (port-)GENERIC systems based on either the entropy or the internal energy formulation of the differential equations. The proposed port-GENERIC formulation distinguishes three kinds of ports, namely reversible ports, irreversible ports with non-negative entropy creation and thermal ports with an indefinite entropy flow. Using a didactic thermo-visco-elastic example, we have shown how subsystems are coupled via these ports: the interconnection of reversible ports yields the canonical Poisson matrix $\mathbf{L}_0$, while closing irreversible ports with unidirectional energy flow or ports at thermal interfaces generates the canonical Onsager matrix $\mathbf{K}_0$. In particular, we have illustrated that the elements of $\mathbf{K}_0$ are directly related to (negative) *modulated resistive* elements of the *entropy bond graph*.

In ongoing and future work, we deal with more complex systems that may feature additional couplings, e.g., dielectric elastomer actuators [27], and the formulation of distributed-parameter port-GENERIC systems with adapted numerics. We will, moreover, highlight the relations to other formulations such as reversible–irreversible port-Hamiltonian systems and extend the framework to account for differential-algebraic equations [28,29] along with structure-preserving time integration [7,16,30]. Understanding port-GENERIC models as *control models*, we will address energy- and entropy-based control designs.

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Appendix A. Basic Bond Graph Elements

Appendix A.1. Junctions

With 0- and 1-junctions, different bonds can be connected in a power-conserving way, generalizing Kirchhoff's laws. At *a* 0-*junction* (corresponding to Kirchhoff's current law), efforts are identical, while the flows sum to zero (we assume here identical orientation of the connected bonds).

$$e_1 = \ldots = e_N, \qquad \sum_{i=1}^{N} f_i = 0. \tag{A1}$$

At a 1-*junction* (corresponding to Kirchhoff's voltage law), the opposite holds:

$$f_1 = \ldots = f_N, \qquad \sum_{i=1}^{N} e_i = 0. \tag{A2}$$

Both types of junctions are illustrated in Figure A1.



Figure A1. Illustration of 0- and 1-junctions connecting 3 bonds with identical orientation. If the orientations of the bonds differ, this must be accounted for by different signs in (A1), (A2).

Appendix A.2. Sources and Causality

Effort and flow sources are denoted Se : e_0 and Sf : f_0. In the case of the effort source, the *causal stroke* at the opposite terminal of the bond represents that the value of the effort signal e_0 is imposed to the subsystem connected there. For flow sources, the causal stroke is drawn at the source, which receives the effort signal from the connected subsystem (and imposes the flow f_0 to this subsystem).



Figure A2. Effort and flow sources with causal strokes.

Appendix A.3. Linear Storage Elements

Because in classical bond graphs physical variables have a unique role as effort or flow (e.g., forces are efforts, velocities are flows), two different energy storage elements are necessary to describe dual forms of stored energy, e.g., kinetic or magnetic vs. potential or electric energy. The former are represented by *I-type* (inductance, inertance), the latter by *C-type* (capacitance, compliance) storage elements. Figure A3 shows, from the mechanical domain, a mass m and a spring with stiffness k (compliance $\frac{1}{k}$). They represent the equations

$$\begin{array}{rcl} \dot{p} & = & F_m \\ v_m & = & \frac{1}{m}p \end{array} \quad \text{and} \quad \begin{array}{rcl} \dot{s} & = & v_s \\ F_s & = & ks \end{array}, \tag{A3}$$

which apparently have the same structure, but reverted roles of the bond variables force F and velocity v. The canonical state variable p denotes the mechanical momentum of the mass, and s the elongation of the spring. The causal strokes in Figure A3 indicate the orientation of the effort signal flow in compliance with the integration of the differential equations in (A3).



Figure A3. Mass and spring as I-type and C-type storage elements with causal strokes according to the *integral causality*.

Appendix A.4. Linear Resistive Elements

Linear resistive elements described by an algebraic relation of type $e = Rf$ are represented by R-elements. Figure A4 shows the bond graph element for a linear damper. Note that no causal stroke is indicated, as the causality depends on whether it is connected in series or in parallel in the considered system.

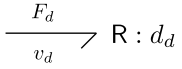

Figure A4. R-element to represent a linear mechanical damper with constitutive relation $F_d = d_d v_d$.

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